Constraint Programming II: Logical Background

François Fages
INRIA Rocquencourt,
Francois.Fages@inria.fr

1. Introduction
2. Logical Background
3. Constraint Logic Programs,
4. Operational and Fixpoint Semantics
5. Logical Semantics, automated deduction
6. Concurrent Constraint Languages
7. Operational and Denotational Semantics
8. Semantics in Linear Logic

Part II: Logical Background

1. First-order formulas
2. Models

   Herbrand's domain, Skolemization
3. Proofs

   Logic programming in *axiomatic theories*

   Decidability of constraint languages in *complete theories*
4. Compactness theorem and Gödel's incompleteness theorem

1. First-Order Terms

Alphabet:

set of variables V,

set of constant and function symbols S_F, given with their arity α

The set T of *first-order terms* is the least set satisfying

i) $V \subset T$

ii) if $f \in S_F$, $\alpha(f) = n$, $M_1, ..., M_n \in T$

then $f(M_1, ..., M_n) \in T$

First-order Formulas

Alphabet: set S_P of predicate symbols.

Atomic propositions: $p(M_1, ..., M_n)$ where $p \in S_P,\ M_1, ..., M_n \in T$.

Formulas: $\neg\phi,\ \phi \vee \psi,\ \exists x\ \phi$

The other logical symbols are defined as abbreviations:

$$
\begin{aligned}
\phi \Rightarrow \psi &= \neg\phi \vee \psi \\
true &= \phi \Rightarrow \phi \\
false &= \neg true \\
\phi \wedge \psi &= \neg(\phi \Rightarrow \neg\psi) \\
\phi \equiv \psi &= (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \\
\forall x \phi &= \neg\exists x \neg\phi
\end{aligned}
$$

Clauses

A *literal* L is either an atomic proposition, A, (called a *positive literal*), or the negation of an atomic proposition, $\neg A$ (called a *negative literal*).

A *clause* is a disjunction of universally quantified literals,

$$\forall(L_1 \vee ... \vee L_n),$$

A *Horn clause* is a clause having at most one positive literal.

$$\neg A_1 \vee ... \vee \neg A_n$$

$$A \vee \neg A_1 \vee ... \vee \neg A_n$$

2. Interpretations

An interpretation $< D, [\,] >$ is a mathematical structure given with

- a domain D,
- distinguished elements $[c] \in D$ for each constant $c \in S_F$,
- operators $[f] : D^n \to D$ for each function symbol $f \in S_F$ of arity n.
- relations $[p] : D^n \to \{true, false\}$ for each predicate symbol $p \in S_P$ of arity n

Valuation

A valuation is a function $\rho : V \to D$ extended to terms by morphism

- $[x]_\rho = \rho(x)$ if $x \in V$,
- $[f(M_1, ..., M_n)]_\rho = [f]([M_1]_\rho, ..., [M_n]_\rho)$ if $f \in S_F$

The truth value of an atom $p(M_1, ..., M_n)$ in an interpretation $I =< D, [\,] >$ and a valuation ρ is the boolean value $[p]([M_1]_\rho, ..., [M_n]_\rho)$.

The truth value of a formula in I and ρ is determined by truth tables and

$[\exists x \phi]_\rho = true$ if $[\phi[d/x]]_\rho = true$ for some $d \in D$, $= false$ otherwise.

$[\forall x \phi]_\rho = true$ if $[\phi[d/x]]_\rho = true$ for every $d \in D$, $= false$ otherwise.

Models

- An interpretation I is a *model* of a closed formula ϕ, $I \models \phi$, if ϕ is true in I.
- A closed formula ϕ' is a *logical consequence* of ϕ closed, $\phi \models \phi'$, if every model of ϕ is a model of ϕ'.
- A formula ϕ is *satisfiable in an interpretation I* if $I \models \exists(\phi)$, ϕ is *valid in I* if $I \models \forall(\phi)$.
- A formula ϕ is *satisfiable* if $\exists(\phi)$ has a model (e.g. $x < 0$)
- A formula is *valid*, noted $\models \phi$, if every interpretation is a model of $\forall(\phi)$ (e.g. $p(x) \Rightarrow \exists y p(y)$)

Proposition 1 *For closed formulas, $\phi \models \phi'$ iff $\models \phi \Rightarrow \phi'$.*

Herbrand's Domain $\mathcal{H}$

Domain of closed terms $T(S_F)$ "Syntactic" interpretation

$[c] = c$

$[f(M_1, ..., M_n)] = f([M_1], ..., [M_n])$

Herbrand's base $B_{\mathcal{H}} = \{p(M_1, ..., M_n) \mid p \in S_P,\ M_i \in T(S_F)\}$

A *Herbrand's interpretation* is identified to a subset of B_H

(the subset defines the atomic propositions which are true).

Herbrand's Models

Proposition 2 *Let S be a set of clauses. S is satisfiable if and only if S has a Herbrand's model.*

Proof: Let us show that if S has a model I then S has a Herbrand model.

For every I-valuation ρ, for every clause $C \in S$, there exists a positive literal A (resp. negative literal $\neg A$) in C such that $I \models A\rho$ (resp. $I \not\models A\rho$).

Let I' be the Herbrand's interpretation defined by

$$I' = \{p(M_1, ..., M_n) \in B_H \mid I \models p(M_1, ..., M_n)\}.$$

For every Herbrand's valuation ρ', for every clause, there exists a literal A (resp. $\neg A$) such that $I' \models A\rho'$ (resp. $I' \not\models A\rho'$).

Therefore I' is a Herbrand's model of S. $\square$

Skolemization

- Put ϕ in prenex form (all quantifiers in the head)
- Replace an existential variable x by a term $f(x_1, ..., x_k)$ where f is a new function symbol and the x_i's are the universal variables before x

E.g. $\phi = \forall x \exists y \forall z\ p(x, y, z)$, $\phi^s = \forall x \forall z\ p(x, f(x), z)$.

Proposition 3 *Any formula ϕ is satisfiable iff its Skolem's normal form ϕ^s is satisfiable.*

Proof: If $I \models \phi$ then one can choose an interpretation of the Skolem's function symbols in ϕ^s according to the I-valuation of the existential variables of ϕ such that $I \models \phi^s$.

Conversely, if $I \models \phi^s$, the interpretation of the Skolem's functions in ϕ^s gives a valuation of the existential variables in ϕ s.t. $I \models \phi$. $\square$

3. Logical Theories

A *theory* is a formal system formed with

- logical axioms and inference rules

$$\neg A \vee A \text{ (excluded middle)} \qquad A[x \leftarrow B] \Rightarrow \exists x\ A \text{ (substitution)},$$

$$\frac{A}{B \vee A} \text{ (Weakening)}, \qquad \frac{A \vee A}{A} \text{ (Contraction)},$$

$$\frac{A \vee (B \vee C)}{(A \vee B) \vee C} \text{ (Associativity)}, \qquad \frac{A \vee B \quad \neg A \vee C}{B \vee C} \text{ (Cut)},$$

$$\frac{A \Rightarrow B \quad x \notin V(B)}{\exists x A \Rightarrow B} \text{ (Existential introduction)}.$$

- a set $\mathcal{T}$ of non-logical axioms

Deduction relation: $\mathcal{T} \vdash \phi$ if the closed formula ϕ can be derived in $\mathcal{T}$

$\mathcal{T}$ is *contradictory* if $\mathcal{T} \vdash false$, otherwise $\mathcal{T}$ is *consistent*.

Validity

Theorem 4 (Deduction theorem) $\mathcal{T} \vdash \phi \Rightarrow \psi$ *iff* $\mathcal{T} \cup \{\phi\} \models \psi$.

Proof: The implication is immediate with the cut rule.

Conversely the proof is by induction on the derivation of the formula ψ. □

Theorem 5 (Validity) *If* $\mathcal{T} \vdash \phi$ *then* $\mathcal{T} \models \phi$.

Proof: By induction on the length of the deduction of ϕ. □

Corollary 6 *If* $\mathcal{T}$ *has a model then* $\mathcal{T}$ *is consistent*

Proof: We show the contrapositive: if $\mathcal{T}$ is contradictory, then $\mathcal{T} \vdash false$, hence $\mathcal{T} \models false$, hence $\mathcal{T}$ has no model. □

4. Gödel's Completeness Theorem

Theorem 7 *A theory is consistent iff it has a model.*

PROOF: The idea is to construct a Herbrand's model of the theory supposed to be consistent, by interpreting by true the closed atoms which are theorems of $\mathcal{T}$, and by false the closed atoms whose negation is a theorem of $\mathcal{T}$. For this it is necessary to extend the alphabet to denote domain elements by Herbrand terms. $\square$

Corollary 8

$$\mathcal{T} \models \phi \textit{ iff } \mathcal{T} \vdash \phi.$$

PROOF: If $\mathcal{T} \models \phi$ then $\mathcal{T} \cup \{\neg\phi\}$ has no model, hence $\mathcal{T} \cup \{\neg\phi\} \vdash false$, and by the deduction theorem $\mathcal{T} \vdash \neg\neg\phi$, now by the cut rule with the axiom of excluded middle (plus weakening and contraction) we get $\mathcal{T} \vdash \phi$. $\square$

Axiomatic and Complete Theories

A theory $\mathcal{T}$ is *axiomatic* if the set of non logical axioms is recursive (i.e. membership to this set can be decided by an algorithm).

Proposition 9 *In an axiomatic theory T, valid formulas, $\mathcal{T} \models \phi$, are recursively enumerable.*

(expresses the feasibility of the Logic Programming paradigm...)

A theory is *complete* if for every closed formula ϕ, either $\mathcal{T} \vdash \phi$ or $\mathcal{T} \vdash \neg\phi$.

In a complete axiomatic theory, we can decide whether an arbitrary formula is satisfiable or not (Constraint Satisfaction paradigm...).

Compactness theorem

Theorem 10 *$\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$.*

PROOF: By Gödel's completeness theorem, $\mathcal{T} \models \phi$ iff $\mathcal{T} \vdash \phi$.

As the proofs are finite, they use only a finite part of non logical axioms $\mathcal{T}$.

Therefore $\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$. $\square$

Corollary 11 *$\mathcal{T}$ is consistent iff every finite part of $\mathcal{T}$ is consistent.*

PROOF: $\mathcal{T}$ has a model iff $\mathcal{T} \models false$,

iff for some finite part $\mathcal{T}'$ of $\mathcal{T}$, $\mathcal{T}' \models false$,

iff some finite part of $\mathcal{T}$ has no model. $\square$

Coloring infinite maps with four colors

Let $\mathcal{T}$ express the coloriability with four colors of an infinite planar graph G:

- $\forall x \ \bigvee_{i=1}^{4} c_i(x)$,
- $\forall x \ \bigwedge_{1\leq i<j\leq 4} \neg(c_i(x) \wedge c_j(x))$,
- $\bigwedge_{i=1}^{4} \neg(c_i(a) \wedge c_i(b))$ for every adjacent vertices a, b in G.

Let $\mathcal{T}'$ be any finite part of $\mathcal{T}$, and G' be the (finite) subgraph of G containing the vertices which appear in $\mathcal{T}'$. As G' is finite and planar it can be colored with 4 colors [Appel and Haken 76], thus $\mathcal{T}'$ has a model.

Now as every finite part $\mathcal{T}'$ of $\mathcal{T}$ is satisfiable, we deduce from the compactness theorem that $\mathcal{T}$ is satisfiable. Therefore every infinite planar graph can be colored with four colors.

Complete theory: Presburger's arithmetic

Complete axiomatic theory of $(\mathbf{N}, 0, s, +, =)$,

E_1 : $\forall x\ x = x$,

E_2 : $\forall x \forall y\ x = y \rightarrow s(x) = s(y)$,

E_3 : $\forall x \forall y \forall z \forall v\ x = y \wedge z = v \rightarrow (x = z \rightarrow y = v)$,

E_4, Π_1: $\forall x \forall y\ s(x) = s(y) \rightarrow x = y$,

E_5, Π_2: $\forall x\ 0 \neq s(x)$,

Π_3: $\forall x\ x + 0 = x$,

Π_4: $\forall x\ x + s(y) = s(x + y)$,

Π_5: $\phi[x \leftarrow 0] \wedge (\forall x\ \phi \rightarrow \phi[x \leftarrow s(x)]) \rightarrow \forall x \phi$ for every formula ϕ.

Note that $E_6 :\ \forall x\ x \neq s(x)$ and $E_7 :\ \forall x\ x = 0 \vee \exists y\ x = s(y)$ are provable by induction.

Gödel's Incompleteness Theorem

Peano's arithmetic contains moreover two axioms for $\times$:

Π_6: $\forall x\ x \times 0 = 0$,

Π_7: $\forall x \forall y\ x \times s(y) = x \times y + x$,

Theorem 12 *Any consistent axiomatic extension of Peano's arithmetic is incomplete.*

Proof: The idea of the proof, following the liar paradox of Epimenides (600 bc) which says: "I lie", is to construct in the language of Peano's arithmetic Π a formula ϕ which is true in the structure of natural numbers $\mathbf{N}$ if and only if ϕ is not provable in Π. As $\mathbf{N}$ is a model of Π, ϕ is necessarily true in $\mathbf{N}$ and not provable in Π, hence Π is incomplete. $\square$

Corollary 13 *The structure* $(\mathcal{N}, 0, 1, +, *)$ *is not axiomatizable.*