Constraint Programming II: Logical Background

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1. First-order formulas
2. Model theory

   Herbrand's domain, Skolemization
3. Proof theory

   Logic programming in *axiomatic theories*

   Decidability of constraint languages in *complete theories*
4. Compactness theorem and Gödel's completeness and incompleteness theorems

1. First-Order Terms

Alphabet:

infinite set of variables V,

set of constant and function symbols S_F, given with their arity α

The set T of first-order terms is the *least* set satisfying

i) $V \subset T$

ii) if $f \in S_F,\ \alpha(f) = n,\ M_1, ..., M_n \in T$
then $f(M_1, ..., M_n) \in T$

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The *principle of structural induction* applies to such *inductive definitions*: a property on terms is true if it is true for variables, and true for terms of the form $f(M_1, ..., M_n)$ supposing it true for $M_1, ..., M_n$.

First-order Formulas

Alphabet: set S_P of predicate symbols.

Atomic propositions: $p(M_1, ..., M_n)$ where $p \in S_P,\ M_1, ..., M_n \in T$.

Formulas: $\neg\phi,\ \phi \vee \psi,\ \exists x\ \phi$

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The other logical symbols are defined as abbreviations:

$$
\begin{array}{rcl}
\phi \Rightarrow \psi & = & \neg\phi \vee \psi \\
true & = & \phi \Rightarrow \phi \\
false & = & \neg true \\
\phi \wedge \psi & = & \neg(\phi \Rightarrow \neg\psi) \\
\phi \equiv \psi & = & (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \\
\forall x\phi & = & \neg\exists x\neg\phi
\end{array}
$$

Clauses

A *literal* L is either an atomic proposition, A, (called a *positive literal*), or the negation of an atomic proposition, $\neg A$ (called a *negative literal*).

A *clause* is a disjunction of universally quantified literals,

$$\forall(L_1 \vee ... \vee L_n),$$

A *Horn clause* is a clause having at most one positive literal.

$$\neg A_1 \vee ... \vee \neg A_n$$

$$A \vee \neg A_1 \vee ... \vee \neg A_n$$

2. Interpretations

An interpretation $< D, [\,] >$ is a mathematical structure given with

- a domain D,
- distinguished elements $[c] \in D$ for each constant $c \in S_F$,
- operators $[f] : D^n \to D$ for each function symbol $f \in S_F$ of arity n.
- relations $[p] : D^n \to \{true, false\}$ for each predicate symbol $p \in S_P$ of arity n

Valuation

A valuation is a function $\rho : V \to D$ extended to terms by morphism

- $[x]_\rho = \rho(x)$ if $x \in V$,
- $[f(M_1, ..., M_n)]_\rho = [f]([M_1]_\rho, ..., [M_n]_\rho)$ if $f \in S_F$

The truth value of an atom $p(M_1, ..., M_n)$ in an interpretation $I =< D, [\,] >$ and a valuation ρ is the boolean value $[p]([M_1]_\rho, ..., [M_n]_\rho)$.

The truth value of a formula in I and ρ is determined by truth tables and

$[\exists x \phi]_\rho = true$ if $[\phi[d/x]]_\rho = true$ for some $d \in D$, $= false$ otherwise.

$[\forall x \phi]_\rho = true$ if $[\phi[d/x]]_\rho = true$ for every $d \in D$, $= false$ otherwise.

Models

- An interpretation I is a *model* of a closed formula ϕ, $I \models \phi$, if ϕ is true in I.
- A closed formula ϕ' is a *logical consequence* of ϕ closed, $\phi \models \phi'$, if every model of ϕ is a model of ϕ'.
- A formula ϕ is *satisfiable in an interpretation I* if $I \models \exists(\phi)$,
  (e.g. $\mathcal{Z} \models \exists x\ x < 0$)
  ϕ is *valid in I* if $I \models \forall(\phi)$.

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  ϕ is *valid in I* if $I \models \forall(\phi)$.
- A formula ϕ is *satisfiable* if $\exists(\phi)$ has a model (e.g. $x < 0$)
- A formula is *valid*, noted $\models \phi$, if every interpretation is a model of $\forall(\phi)$ (e.g. $p(x) \Rightarrow \exists y p(y)$)

Proposition 1 *For closed formulas,* $\phi \models \phi'$ *iff* $\models \phi \Rightarrow \phi'$.

Herbrand's Domain $\mathcal{H}$

Domain of closed terms $T(S_F)$ "Syntactic" interpretation

$[c] = c$

$[f(M_1, ..., M_n)] = f([M_1], ..., [M_n])$

Herbrand's base $B_{\mathcal{H}} = \{p(M_1, ..., M_n) \mid p \in S_P,\ M_i \in T(S_F)\}$

A *Herbrand's interpretation* is identified to a subset of B_H

(the subset defines the atomic propositions which are true).

Herbrand's Models

Proposition 2 *Let S be a set of clauses. S is satisfiable if and only if S has a Herbrand's model.*

PROOF: Suppose I is a model of S: for every I-valuation ρ, for every clause $C \in S$, there exists a positive literal A (resp. negative literal $\neg A$) in C such that $I \models A\rho$ (resp. $I \not\models A\rho$).

... $\square$

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Let I' be the Herbrand's interpretation defined by

$$I' = \{p(M_1, ..., M_n) \in B_H \mid I \models p(M_1, ..., M_n)\}.$$

For every Herbrand's valuation ρ', there exists an I-valuation ρ such that $I \models A\rho$ iff $I' \models A\rho'$. Hence, for every clause, there exists a literal A (resp. $\neg A$) such that $I' \models A\rho'$ (resp. $I' \not\models A\rho'$).

Therefore I' is a Herbrand's model of S. □

Satisfiability of Non-Clausal Formula by Skolemization

- Put ϕ in prenex form (all quantifiers in the head)
- Replace an existential variable x by a term $f(x_1, ..., x_k)$ where f is a *new function symbol* and the x_i's are the universal variables before x

E.g. $\phi = \forall x \exists y \forall z \; p(x, y, z)$, $\phi^s = \forall x \forall z \; p(x, f(x), z)$.

Proposition 4 *Any formula ϕ is satisfiable iff its Skolem's normal form ϕ^s is satisfiable.*

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PROOF: If $I \models \phi$ then one can choose an interpretation of the Skolem's function symbols in ϕ^s according to the I-valuation of the existential variables of ϕ such that $I \models \phi^s$.

Conversely, if $I \models \phi^s$, the interpretation of the Skolem's functions in ϕ^s gives a valuation of the existential variables in ϕ s.t. $I \models \phi$. □

3. Logical Theories

A *theory* is a formal system formed with

- logical axioms and inference rules

  $\neg A \vee A$ (excluded middle) $\quad A[x \leftarrow B] \Rightarrow \exists x\ A$ (substitution),

  $$\frac{A}{B \vee A} \text{ (Weakening)}, \qquad \frac{A \vee A}{A} \text{ (Contraction)},$$

  $$\frac{A \vee (B \vee C)}{(A \vee B) \vee C} \text{ (Associativity)}, \qquad \frac{A \vee B \quad \neg A \vee C}{B \vee C} \text{ (Cut)},$$

  $$\frac{A \Rightarrow B \quad x \notin V(B)}{\exists x A \Rightarrow B} \text{ (Existential introduction).}$$

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Deduction relation: $\mathcal{T} \vdash \phi$ if the closed formula ϕ can be derived in $\mathcal{T}$

$\mathcal{T}$ is *contradictory* if $\mathcal{T} \vdash false$, otherwise $\mathcal{T}$ is *consistent*.

Deduction Theorem

Theorem 6 $\mathcal{T} \vdash \phi \Rightarrow \psi$ *iff* $\mathcal{T} \cup \{\phi\} \vdash \psi$.

PROOF: The implication is immediate with the cut rule.

Conversely the proof is by structural induction on the derivation of the formula ψ. □

Validity Theorem

Theorem 7 *If* $\mathcal{T} \vdash \phi$ *then* $\mathcal{T} \models \phi$.

PROOF: By induction on the length of the deduction of ϕ. □

Corollary 8 *If* $\mathcal{T}$ *has a model then* $\mathcal{T}$ *is consistent*

PROOF: We show the contrapositive: if $\mathcal{T}$ is contradictory, then $\mathcal{T} \vdash false$, hence $\mathcal{T} \models false$, hence $\mathcal{T}$ has no model. □

4. Gödel's Completeness Theorem

Theorem 9 *A theory is consistent iff it has a model.*

PROOF: Supposing the theory consistent, the idea is to construct a Herbrand's model of the theory, by interpreting by true the closed atoms which are theorems of $\mathcal{T}$, and by false the closed atoms whose negation is a theorem of $\mathcal{T}$.

For this it is necessary to extend the alphabet to denote domain elements by Herbrand terms. □

Corollary 10 $\mathcal{T} \models \phi$ *iff* $\mathcal{T} \vdash \phi$.

PROOF: If $\mathcal{T} \models \phi$ then $\mathcal{T} \cup \{\neg\phi\}$ has no model, hence $\mathcal{T} \cup \{\neg\phi\} \vdash false$, and by the deduction theorem $\mathcal{T} \vdash \neg\neg\phi$, now by the cut rule with the axiom of excluded middle (plus weakening and contraction) we get $\mathcal{T} \vdash \phi$. □

Axiomatic and Complete Theories

A theory $\mathcal{T}$ is *axiomatic* if the set of non logical axioms is recursive (i.e. membership to this set can be decided by an algorithm).

Proposition 11 *In an axiomatic theory T, valid formulas, $\mathcal{T} \models \phi$, are recursively enumerable.*

(expresses the feasibility of the Logic Programming paradigm...)

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A theory is *complete* if for every closed formula ϕ, either $\mathcal{T} \vdash \phi$ or $\mathcal{T} \vdash \neg\phi$.

In a complete axiomatic theory, we can decide whether an arbitrary formula is satisfiable or not (Constraint Satisfaction paradigm...).

Compactness theorem

Theorem 13 *$\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$.*

PROOF: By Gödel's completeness theorem, $\mathcal{T} \models \phi$ iff $\mathcal{T} \vdash \phi$.

As the proofs are finite, they use only a finite part of non logical axioms $\mathcal{T}$.

Therefore $\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$. □

Compactness theorem

Theorem 14 *$\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$.*

Proof: By Gödel's completeness theorem, $\mathcal{T} \models \phi$ iff $\mathcal{T} \vdash \phi$.

As the proofs are finite, they use only a finite part of non logical axioms $\mathcal{T}$.

Therefore $\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$. □

Corollary 15 *$\mathcal{T}$ is consistent iff every finite part of $\mathcal{T}$ is consistent.*

Proof: $\mathcal{T}$ is inconsistent iff $\mathcal{T} \vdash false$,

iff for some finite part $\mathcal{T}'$ of $\mathcal{T}$, $\mathcal{T}' \vdash false$,

iff some finite part of $\mathcal{T}$ is inconsistent. □

Coloring infinite maps with four colors

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Let $\mathcal{T}'$ be any finite part of $\mathcal{T}$, and G' be the (finite) subgraph of G containing the vertices which appear in $\mathcal{T}'$. As G' is finite and planar it can be colored with 4 colors [Appel and Haken 76], thus $\mathcal{T}'$ has a model.

Now as every finite part $\mathcal{T}'$ of $\mathcal{T}$ is satisfiable, we deduce from the compactness theorem that $\mathcal{T}$ is satisfiable. Therefore every infinite planar graph can be colored with four colors.

Complete theory: Presburger's arithmetic

Complete axiomatic theory of $(\mathbf{N}, 0, s, +, =)$,

E_1 : $\forall x\ x = x$,

E_2 : $\forall x \forall y\ x = y \rightarrow s(x) = s(y)$,

E_3 : $\forall x \forall y \forall z \forall v\ x = y \wedge z = v \rightarrow (x = z \rightarrow y = v)$,

E_4, Π_1: $\forall x \forall y\ s(x) = s(y) \rightarrow x = y$,

E_5, Π_2: $\forall x\ 0 \neq s(x)$,

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Π_3: $\forall x\ x + 0 = x$,

Π_4: $\forall x\ x + s(y) = s(x + y)$,

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$\Pi_3:\ \forall x\ x + 0 = x,$

$\Pi_4:\ \forall x\ x + s(y) = s(x + y),$

$\Pi_5:\ \phi[x \leftarrow 0] \wedge (\forall x\ \phi \rightarrow \phi[x \leftarrow s(x)]) \rightarrow \forall x \phi$ for every formula ϕ.

Note that $E_6 :\ \forall x\ x \neq s(x)$ and $E_7 :\ \forall x\ x = 0 \vee \exists y\ x = s(y)$ are provable by induction.

Incomplete Theory: Peano's arithmetic

Peano's arithmetic contains moreover two axioms for $\times$:

Π_6: $\forall x\ x \times 0 = 0,$

Π_7: $\forall x \forall y\ x \times s(y) = x \times y + x,$

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Theorem 16 (Gödel's Incompleteness Theorem) *Any consistent axiomatic extension of Peano's arithmetic is incomplete.*

PROOF: The idea of the proof, following the liar paradox of Epimenides (600 bc) which says: "I lie", is to construct in the language of Peano's arithmetic Π a formula ϕ which is true in the structure of natural numbers $\mathbf{N}$ if and only if ϕ is not provable in Π. As $\mathbf{N}$ is a model of Π, ϕ is necessarily true in $\mathbf{N}$ and not provable in Π, hence Π is incomplete. $\square$

Corollary 17 *The structure* $(\mathcal{N}, 0, 1, +, *)$ *is not axiomatizable.*