

Constraint Programming II: Logical Background

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1. First-order formulas

2. Model theory

Herbrand's domain, Skolemization

3. Proof theory

Logic programming in *axiomatic theories*

Decidability of constraint languages in *complete theories*

4. Compactness theorem and Gödel's completeness and incompleteness theorems

1. First-Order Terms

Alphabet:

infinite set of variables V ,

set of constant and function symbols S_F , given with their arity α

The set T of first-order terms is the *least* set satisfying

i) $V \subset T$

ii) if $f \in S_F$, $\alpha(f) = n$, $M_1, \dots, M_n \in T$
then $f(M_1, \dots, M_n) \in T$

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The *principle of structural induction* applies to such *inductive definitions*:
a property on terms is true if it is true for variables, and true for terms of
the form $f(M_1, \dots, M_n)$ supposing it true for M_1, \dots, M_n .

First-order Formulas

Alphabet: set S_P of **predicate symbols**.

Atomic propositions: $p(M_1, \dots, M_n)$ where $p \in S_P$, $M_1, \dots, M_n \in T$.

Formulas: $\neg\phi$, $\phi \vee \psi$, $\exists x \phi$

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The other logical symbols are defined as abbreviations:

$$\phi \Rightarrow \psi = \neg\phi \vee \psi$$

$$true = \phi \Rightarrow \phi$$

$$false = \neg true$$

$$\phi \wedge \psi = \neg(\phi \Rightarrow \neg\psi)$$

$$\phi \equiv \psi = (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$$

$$\forall x \phi = \neg \exists x \neg \phi$$

Clauses

A *literal* L is either an atomic proposition, A , (called a *positive literal*), or the negation of an atomic proposition, $\neg A$ (called a *negative literal*).

A *clause* is a disjunction of universally quantified literals,

$$\forall(L_1 \vee \dots \vee L_n),$$

A *Horn clause* is a clause having at most one positive literal.

$$\neg A_1 \vee \dots \vee \neg A_n$$

$$A \vee \neg A_1 \vee \dots \vee \neg A_n$$

2. Interpretations

An interpretation $\langle D, [\] \rangle$ is a mathematical structure given with

- a domain D ,
- distinguished elements $[c] \in D$ for each constant $c \in S_F$,
- operators $[f] : D^n \rightarrow D$ for each function symbol $f \in S_F$ of arity n .
- relations $[p] : D^n \rightarrow \{true, false\}$ for each predicate symbol $p \in S_P$ of arity n

Valuation

A **valuation** is a function $\rho : V \rightarrow D$ extended to terms by morphism

- $[x]_\rho = \rho(x)$ if $x \in V$,
- $[f(M_1, \dots, M_n)]_\rho = [f]([M_1]_\rho, \dots, [M_n]_\rho)$ if $f \in S_F$

The **truth value of an atom** $p(M_1, \dots, M_n)$ in an interpretation $I = \langle D, [] \rangle$ and a valuation ρ is the boolean value $[p]([M_1]_\rho, \dots, [M_n]_\rho)$.

The **truth value of a formula** in I and ρ is determined by truth tables and

$[\exists x \phi]_\rho = \text{true}$ if $[\phi[d/x]]_\rho = \text{true}$ for some $d \in D$, = *false* otherwise.

$[\forall x \phi]_\rho = \text{true}$ if $[\phi[d/x]]_\rho = \text{true}$ for every $d \in D$, = *false* otherwise.

Models

- An interpretation I is a *model* of a closed formula ϕ , $I \models \phi$, if ϕ is true in I .
- A closed formula ϕ' is a *logical consequence* of ϕ closed, $\phi \models \phi'$, if every model of ϕ is a model of ϕ' .
- A formula ϕ is *satisfiable in an interpretation I* if $I \models \exists(\phi)$,
(e.g. $\mathcal{Z} \models \exists x x < 0$)
 ϕ is *valid in I* if $I \models \forall(\phi)$.

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 ϕ is *valid in I* if $I \models \forall(\phi)$.
- A formula ϕ is *satisfiable* if $\exists(\phi)$ has a model (e.g. $x < 0$)
- A formula is *valid*, noted $\models \phi$,
if every interpretation is a model of $\forall(\phi)$ (e.g. $p(x) \Rightarrow \exists y p(y)$)

Proposition 1 For closed formulas, $\phi \models \phi'$ iff $\models \phi \Rightarrow \phi'$.

Herbrand's Domain \mathcal{H}

Domain of closed terms $T(S_F)$ “Syntactic” interpretation

$$[c] = c$$

$$[f(M_1, \dots, M_n)] = f([M_1], \dots, [M_n])$$

Herbrand's base $B_{\mathcal{H}} = \{p(M_1, \dots, M_n) \mid p \in S_P, M_i \in T(S_F)\}$

A *Herbrand's interpretation* is identified to a subset of B_H
(the subset defines the atomic propositions which are true).

Herbrand's Models

Proposition 2 *Let S be a set of clauses. S is *satisfiable* if and only if S has a *Herbrand's model*.*

PROOF: Suppose I is a model of S : for every I -valuation ρ , for every clause $C \in S$, there exists a positive literal A (resp. negative literal $\neg A$) in C such that $I \models A\rho$ (resp. $I \not\models A\rho$).

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Let I' be the Herbrand's interpretation defined by

$$I' = \{p(M_1, \dots, M_n) \in B_H \mid I \models p(M_1, \dots, M_n)\}.$$

For every Herbrand's valuation ρ' , there exists an I -valuation ρ such that $I \models A\rho$ iff $I' \models A\rho'$. Hence, for every clause, there exists a literal A (resp. $\neg A$) such that $I' \models A\rho'$ (resp. $I' \not\models A\rho'$).

Therefore I' is a Herbrand's model of S . □

Satisfiability of Non-Clausal Formula by Skolemization

- Put ϕ in prenex form (all quantifiers in the head)
- Replace an existential variable x by a term $f(x_1, \dots, x_k)$ where f is a *new function symbol* and the x_i 's are the universal variables before x

E.g. $\phi = \forall x \exists y \forall z p(x, y, z)$, $\phi^s = \forall x \forall z p(x, f(x), z)$.

Proposition 4 *Any formula ϕ is satisfiable iff its Skolem's normal form ϕ^s is satisfiable.*

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PROOF: If $I \models \phi$ then one can choose an interpretation of the Skolem's function symbols in ϕ^s according to the I -valuation of the existential variables of ϕ such that $I \models \phi^s$.

Conversely, if $I \models \phi^s$, the interpretation of the Skolem's functions in ϕ^s gives a valuation of the existential variables in ϕ s.t. $I \models \phi$. □

3. Logical Theories

A *theory* is a formal system formed with

- logical axioms and inference rules

$\neg A \vee A$ (excluded middle)

$\frac{A}{B \vee A}$ (Weakening),

$\frac{A \vee (B \vee C)}{(A \vee B) \vee C}$ (Associativity),

$\frac{A \Rightarrow B \quad x \notin V(B)}{\exists x A \Rightarrow B}$ (Existential introduction).

$A[x \leftarrow B] \Rightarrow \exists x A$ (substitution),

$\frac{A \vee A}{A}$ (Contraction),

$\frac{A \vee B \quad \neg A \vee C}{B \vee C}$ (Cut),

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Deduction relation: $\mathcal{T} \vdash \phi$ if the closed formula ϕ can be derived in \mathcal{T}

\mathcal{T} is *contradictory* if $\mathcal{T} \vdash \text{false}$, otherwise \mathcal{T} is *consistent*.

Deduction Theorem

Theorem 6 $\mathcal{T} \vdash \phi \Rightarrow \psi$ iff $\mathcal{T} \cup \{\phi\} \vdash \psi$.

PROOF: The implication is immediate with the cut rule.

Conversely the proof is by structural induction on the derivation of the formula ψ . □

Validity Theorem

Theorem 7 *If $\mathcal{T} \vdash \phi$ then $\mathcal{T} \models \phi$.*

PROOF: By induction on the length of the deduction of ϕ . □

Corollary 8 *If \mathcal{T} has a model then \mathcal{T} is consistent*

PROOF: We show the contrapositive: if \mathcal{T} is contradictory, then $\mathcal{T} \vdash \text{false}$, hence $\mathcal{T} \models \text{false}$, hence \mathcal{T} has no model. □

4. Gödel's Completeness Theorem

Theorem 9 *A theory is consistent iff it has a model.*

PROOF: Supposing the theory consistent, the idea is to construct a Herbrand's model of the theory, by interpreting by true the closed atoms which are theorems of \mathcal{T} , and by false the closed atoms whose negation is a theorem of \mathcal{T} .

For this it is necessary to extend the alphabet to denote domain elements by Herbrand terms. □

Corollary 10 $\mathcal{T} \models \phi$ iff $\mathcal{T} \vdash \phi$.

PROOF: If $\mathcal{T} \models \phi$ then $\mathcal{T} \cup \{\neg\phi\}$ has no model, hence $\mathcal{T} \cup \{\neg\phi\} \vdash \text{false}$, and by the deduction theorem $\mathcal{T} \vdash \neg\neg\phi$, now by the cut rule with the axiom of excluded middle (plus weakening and contraction) we get $\mathcal{T} \vdash \phi$. □

Axiomatic and Complete Theories

A theory \mathcal{T} is *axiomatic* if the set of non logical axioms is recursive (i.e. membership to this set can be decided by an algorithm).

Proposition 11 *In an axiomatic theory T , valid formulas, $\mathcal{T} \models \phi$, are recursively enumerable.*

(expresses the feasibility of the **Logic Programming paradigm...**)

Axiomatic and Complete Theories

A theory \mathcal{T} is *axiomatic* if the set of non logical axioms is recursive (i.e. membership to this set can be decided by an algorithm).

Proposition 12 *In an axiomatic theory T , valid formulas, $\mathcal{T} \models \phi$, are recursively enumerable.*

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A theory is *complete* if for every closed formula ϕ , either $\mathcal{T} \vdash \phi$ or $\mathcal{T} \vdash \neg\phi$.

In a **complete axiomatic theory**, we can decide whether an arbitrary formula is satisfiable or not (**Constraint Satisfaction paradigm...**).

Compactness theorem

Theorem 13 $\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part \mathcal{T}' of \mathcal{T} .

PROOF: By Gödel's completeness theorem, $\mathcal{T} \models \phi$ iff $\mathcal{T} \vdash \phi$.

As the proofs are finite, they use only a finite part of non logical axioms \mathcal{T} .

Therefore $\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part \mathcal{T}' of \mathcal{T} . □

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Therefore $\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part \mathcal{T}' of \mathcal{T} . □

Corollary 15 \mathcal{T} is consistent iff every finite part of \mathcal{T} is consistent.

PROOF: \mathcal{T} is inconsistent iff $\mathcal{T} \vdash \text{false}$,

iff for some finite part \mathcal{T}' of \mathcal{T} , $\mathcal{T}' \vdash \text{false}$,

iff some finite part of \mathcal{T} is inconsistent. □

Coloring infinite maps with four colors

Let \mathcal{T} express the [colorability with four colors](#) of an infinite planar graph G :

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Let \mathcal{T}' be any finite part of \mathcal{T} , and G' be the (finite) subgraph of G containing the vertices which appear in \mathcal{T}' . As G' is finite and planar it can be colored with 4 colors [Appel and Haken 76], thus \mathcal{T}' has a model.

Now as every finite part \mathcal{T}' of \mathcal{T} is satisfiable, we deduce from the compactness theorem that [\$\mathcal{T}\$ is satisfiable](#). Therefore every infinite planar graph can be colored with four colors.

Complete theory: Presburger's arithmetic

Complete axiomatic theory of $(\mathbf{N}, 0, s, +, =)$,

$$E_1 : \forall x \ x = x,$$

$$E_2 : \forall x \forall y \ x = y \rightarrow s(x) = s(y),$$

$$E_3 : \forall x \forall y \forall z \forall v \ x = y \wedge z = v \rightarrow (x = z \rightarrow y = v),$$

$$E_4, \Pi_1 : \forall x \forall y \ s(x) = s(y) \rightarrow x = y,$$

$$E_5, \Pi_2 : \forall x \ 0 \neq s(x),$$

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$$\Pi_4 : \forall x \ x + s(y) = s(x + y),$$

$$\Pi_5 : \phi[x \leftarrow 0] \wedge (\forall x \ \phi \rightarrow \phi[x \leftarrow s(x)]) \rightarrow \forall x \phi \text{ for every formula } \phi.$$

Note that $E_6 : \forall x \ x \neq s(x)$ and $E_7 : \forall x \ x = 0 \vee \exists y \ x = s(y)$ are provable by induction.

Incomplete Theory: Peano's arithmetic

Peano's arithmetic contains moreover two axioms for \times :

$$\Pi_6: \quad \forall x \quad x \times 0 = 0,$$

$$\Pi_7: \quad \forall x \forall y \quad x \times s(y) = x \times y + x,$$

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Theorem 16 (Gödel's Incompleteness Theorem) *Any consistent axiomatic extension of Peano's arithmetic is incomplete.*

PROOF: The idea of the proof, following the liar paradox of Epimenides (600 bc) which says: "I lie", is to construct in the language of Peano's arithmetic Π a formula ϕ which is true in the structure of natural numbers \mathbf{N} if and only if ϕ is not provable in Π . As \mathbf{N} is a model of Π , ϕ is necessarily true in \mathbf{N} and not provable in Π , hence Π is incomplete. \square

Corollary 17 *The structure $(\mathcal{N}, 0, 1, +, *)$ is not axiomatizable.*