Constraint Logic Programming

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Part I: CLP - Introduction and Logical Background

Axiomatic and Complete Theories

A theory $\mathcal{T}$ is *axiomatic* if the set of non logical axioms is recursive (i.e., membership can be decided by an algorithm)

Proposition 1

In an axiomatic theory T, valid formulas, $\mathcal{T} \models \phi$, are recursively enumerable

(feasibility of the Logic Programming paradigm...)

$\mathcal{T}$ is *complete* if for every closed ϕ, either $\mathcal{T} \vdash \phi$ or $\mathcal{T} \vdash \neg\phi$

In a complete axiomatic theory, we can decide whether an arbitrary formula is satisfiable or not (Constraint Satisfaction paradigm...)

Compactness theorem

Theorem 2

Corollary 3

$\mathcal{T}$ is consistent iff every finite part of $\mathcal{T}$ is consistent.

$\mathcal{T}$ is inconsistent iff $\mathcal{T} \vdash$ false,
iff for some finite part $\mathcal{T}'$ of $\mathcal{T}$, $\mathcal{T}' \vdash$ false,
iff some finite part of $\mathcal{T}$ is inconsistent $\square$

Compactness theorem

Theorem 2

$\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$

By Gödel's completeness theorem, $\mathcal{T} \models \phi$ iff $\mathcal{T} \vdash \phi$.
As the proofs are finite, they use only a finite part of non logical axioms $\mathcal{T}$.
Therefore $\mathcal{T} \models \phi$ iff $\mathcal{T}' \models \phi$ for some finite part $\mathcal{T}'$ of $\mathcal{T}$ $\square$

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iff some finite part of $\mathcal{T}$ is inconsistent $\square$

Part II: Constraint Logic Programs

6. Constraint Languages
7. CLP($\mathcal{X}$)
8. CLP($\mathcal{H}$)
9. CLP($\mathcal{R}, \mathcal{FD}, \mathcal{B}$)

Part III

CLP - Operational and Fixpoint Semantics

Part III: CLP - Operational and Fixpoint Semantics

Operational semantics: CSLD Resolution

A CLP($\mathcal{X}$) program *P* is a set of clauses representing inductive definitions of constraints. Taking the solver as a black-box a CSLD resolution step is:

A successful derivation is a derivation of the form

$$G \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \ldots \longrightarrow c|\square$$

c is called a for *G*

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$$\frac{}{(c|\alpha, p(s_1, s_2), \alpha') \longrightarrow}$$

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$$\frac{(p(t_1, t_2) \leftarrow c' | A_1, \ldots, A_n)\theta \in P}{(c | \alpha, p(s_1, s_2), \alpha') \longrightarrow}$$

where θ is a renaming substitution of the program clause with new variables

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$$\frac{(p(t_1, t_2) \leftarrow c' | A_1, \ldots, A_n)\theta \in P \quad \mathcal{X} \models \exists (c \wedge s_1 = t_1 \wedge s_2 = t_2 \wedge c')}{(c | \alpha, p(s_1, s_2), \alpha') \longrightarrow (c, s_1 = t_1, s_2 = t_2, c' \mid \alpha, A_1, \ldots, A_n, \alpha')}$$

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A successful derivation is a derivation of the form

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where θ is a renaming substitution of the program clause with new variables

A successful derivation is a derivation of the form

$$G \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \ldots \longrightarrow c|\Box$$

c is called a computed answer constraint for G

$\wedge$-Compositionality of CSLD-derivations

Lemma 4 ($\wedge$-compositionality)

c is a computed answer for the goal $(d|A_1, \ldots, A_n)$
iff
there exist computed answers $c_1, \ldots, c_n$ *for the goals* $true|A_1, \ldots, true|A_n$*, such that* $c = d \wedge \bigwedge_{i=1}^{n} c_i$ *is satisfiable.*

Corollary 5

Independence of the selection strategy

$\wedge$-Compositionality of CSLD-derivations

Proof.

$(\Leftarrow)$ $d|A_1, \ldots, A_n \rightarrow^*$

$\wedge$-Compositionality of CSLD-derivations

Proof.

$(\Leftarrow)$ $d|A_1, \ldots, A_n \rightarrow^* d \wedge c_1|A_2, \ldots, A_n \cdots \rightarrow^* d \wedge c_1 \wedge \cdots \wedge c_n|\Box$.

$\wedge$-Compositionality of CSLD-derivations

Proof.

($\Leftarrow$) $d|A_1, \ldots, A_n \rightarrow^* d \wedge c_1|A_2, \ldots, A_n \cdots \rightarrow^* d \wedge c_1 \wedge \cdots \wedge c_n|\Box$.
($\Rightarrow$) By induction on the length l of the derivation

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Proof.

($\Leftarrow$) $d|A_1, \ldots, A_n \rightarrow^* d \wedge c_1|A_2, \ldots, A_n \cdots \rightarrow^* d \wedge c_1 \wedge \cdots \wedge c_n|\Box$.
($\Rightarrow$) By induction on the length l of the derivation
If $l = 1$ we have $true|A_1 \rightarrow c_1|\Box$

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Proof.

($\Leftarrow$) $d|A_1, \ldots, A_n \rightarrow^* d \wedge c_1|A_2, \ldots, A_n \cdots \rightarrow^* d \wedge c_1 \wedge \cdots \wedge c_n|\Box$.
($\Rightarrow$) By induction on the length l of the derivation
If $l = 1$ we have $true|A_1 \rightarrow c_1|\Box$
Otherwise, suppose A_1 is the selected atom, there exists a rule $(A_1 \leftarrow d_1|B_1, \ldots, B_k) \in P$ such that
$d|A_1, \ldots, A_n \rightarrow d \wedge d_1|B_1, \ldots, B_k, A_2, \ldots, A_n \rightarrow^* c|\Box$
By induction, there exist computed answers $e_1, \ldots, e_k, c_2, \ldots, c_n$ for the goals $B_1, \ldots, B_k, A_2, \ldots, A_n$ such that
$c = d \wedge d_1 \wedge \bigwedge_{i=1}^{k} e_i \wedge \bigwedge_{j=2}^{n} c_j$. Now let $c_1 = d_1 \wedge \bigwedge_{i=1}^{k} e_i$, c_1 is a computed answer for $true|A_1$ $\Box$

Operational Semantics of CLP($\mathcal{X}$) Programs

Observation of the sets of projected computed answer constraints

$$O(P) = \{(\exists X\ c)|A\ :\ true|A \longrightarrow^* c|\Box,\ \mathcal{X} \models \exists(c),\ X = V(c) \setminus V(A)\}$$

Program equivalence: $P \equiv P'$ iff $O(P) = O(P')$ iff for every goal G, P and P' have same sets of computed answer constraints

Finer observables:
multisets of computed answer constraints
sets of successful CSLD derivations (equivalence of traces)

More abstract observable:
sets of goals having a success
(theorem proving versus programming point of view)

Operational Semantics of CLP($\mathcal{X}$) Programs

Observation of computed answer constraints

$$O_{ca}(P) = \{c|A \ : \ true|A \longrightarrow^* c|\Box, \ \mathcal{X} \models \exists(c)\}$$

$P \equiv_{ca} P'$ iff for every goal G, P and P' have the same sets of computed answer constraints

Observation of ground successes

$$O_{gs}(P) = \{A\rho \in B_{\mathcal{X}} \ : \ true|A \longrightarrow^* c|\Box, \ \mathcal{X} \models c\rho\}$$

$P \equiv_{gs} P'$ iff P and P' have the same ground success sets, iff for every goal G, G has a CSLD refutation in P iff G has one in P'

Some definitions

Let $(S, \leq)$ be a partial order Let $X \subset S$ be a subset of S

- An upper bound of X is an element $a \in S$ such that $\forall x \in X \; x \leq a$
- The maximum element of X, if it exists, is the unique upper bound of X belonging to X
- The least upper bound (lub) of X, if it exists, is the minimum of the upper bounds of X
- A sup-semi-lattice is a partial order such that every finite part admits a lub
- A lattice is a sup-semi-lattice and an inf-semi-lattice
- A chain is an increasing sequence $x_1 \leq x_2 \leq \ldots$
- A partial order is complete if every chain admits a lub
- A function $f : S \rightarrow S$ is monotonic if $x \leq y \Rightarrow f(x) \leq f(y)$
- f is continuous if $f(lub(X)) = lub(f(X))$ for every chain X

Fixpoint theorems

Theorem 6 (Knaster-Tarski)

Let $(S, \leq)$ be a complete partial order, and $f : S \to S$ a continuous operator over S
Then f admits a least fixed point $lfp(f) = f \uparrow \omega$

Proof.

First,

a $= f \uparrow \omega$.

a is a fixed point of f

Let e be any fixed point of f.

hence $a \leq e$ □

Fixpoint theorems

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Proof.

First, as f is continuous, f is monotonic, hence
$\bot \leq f(\bot) \leq f(f(\bot)) \leq \dots$ forms an increasing chain.
Let $a = lub(\{f^n(\bot) \mid n \in \mathbb{N}\}) = f \uparrow \omega$. By continuity
$f(a) = lub(\{f^{n+1}(\bot) \mid n \in \mathbb{N}\}) = a$, hence a is a fixed point of f

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Let e be any fixed point of f. We show that for all integer n, $f^n(\perp) \leq e$, by induction on n.

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Let $a = lub(\{f^n(\bot) \mid n \in \mathbb{N}\}) = f \uparrow \omega$. By continuity
$f(a) = lub(\{f^{n+1}(\bot) \mid n \in \mathbb{N}\}) = a$, hence a is a fixed point of f

Let e be any fixed point of f. We show that for all integer n,
$f^n(\bot) \leq e$, by induction on n. Clearly $\bot \leq e$. Furthermore if
$f^n(\bot) \leq e$ then by monotonicity, $f^{n+1}(\bot) \leq f(e) = e$.
Thus $f^n(\bot) \leq e$ for all n, hence $a \leq e$ □

Least Post-Fixed Point

Theorem 7

Let $(S, \leq)$ be a complete sup-semi-lattice. Let f be a continuous operator over S. Then f admits a least post-fixed point (i.e., an element e satisfying $f(e) \leq e$) which is equal to $lfp(f)$.

Proof.

Least Post-Fixed Point

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Let $g(x) = lub(x, f(x))$.

Least Post-Fixed Point

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Proof.

Let $g(x) = lub(x, f(x))$.
An element e is a post fixed point of f, i.e., $f(e) \leq e$, iff e is a fixed point of g, $g(e) = e$.
Now g is continuous, hence $lfp(g)$ is the least fixed point of g and the least post-fixed point of f.
Furthermore, $lfp(g) = lub\{f^n(\perp)\} = lfp(f)$. □

Fixpoint semantics of O_{gs}

Consider the complete lattice of $\mathcal{X}$-interpretations $(2^{\mathcal{B}_{\mathcal{X}}}, \subset)$
The bottom element is the empty $\mathcal{X}$-interpretation (all atoms false)
The top element is $\mathcal{B}_{\mathcal{X}}$ (all atoms true)

A chain X is an increasing sequence $I_1 \subset I_2 \subset \ldots$
$lub(X) = \bigcup_{i \geq 1} I_i$

Let us define the semantics $O_{gs}(P)$ as the least solution of a fixpoint equation over $2^{\mathcal{B}_{\mathcal{X}}}$: $I = T(I)$

$T_P^{\mathcal{X}}$ immediate consequence operator

$T_P^{\mathcal{X}} : 2^{\mathcal{B}_{\mathcal{X}}} \to 2^{\mathcal{B}_{\mathcal{X}}}$ is defined by:

$T_P^{\mathcal{X}}(I) = \{A\rho \in \mathcal{B}_{\mathcal{X}} |$ there exists a renamed clause in normal form $(A \leftarrow c | A_1, \ldots, A_n) \in P$, and a valuation ρ s.t. $\mathcal{X} \models c\rho$ and $\{A_1\rho, \ldots, A_n\rho\} \subset I\}$

```
append(A,B,C):- A=[], B=C.
append(A,B,C):- A=[X|L], C=[X|R], append(L,B,R).
```

Example 8

$T_P^{\mathcal{H}}(\emptyset) =$

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$T_P^{\mathcal{H}}(\emptyset) = \{append([], B, B) \mid B \in \mathcal{H}\}$

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$$T_P^{\mathcal{H}}(T_P^{\mathcal{H}}(T_P^{\mathcal{H}}(\emptyset))) =$$

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T_P^{\mathcal{H}}(T_P^{\mathcal{H}}(T_P^{\mathcal{H}}(\emptyset))) & = & T_P^{\mathcal{H}}(T_P^{\mathcal{H}}(\emptyset)) \cup \\
 & & \{append([X, Y], B, [X, Y|B]) \mid X, Y, B \in \mathcal{H}\}
\end{array}
$$

Continuity of $T_P^{\mathcal{X}}$ operator

Proposition 9

$T_P^{\mathcal{X}}$ is a continuous operator on the complete lattice of $\mathcal{X}$-interpretations

Proof.

□

Corollary 10

$T_P^{\mathcal{X}}$ admits a least (post) fixed point $T_P^{\mathcal{X}} \uparrow \omega$

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Proof.

Let X be a chain of $\mathcal{X}$-interpretations. $A\rho \in T_P^{\mathcal{X}}(lub(X))$,
iff $(A \leftarrow c|A_1, \ldots, A_n) \in P$, $\mathcal{X} \models c\rho$ and $\{A_1\rho, \ldots, A_n\rho\} \subset lub(X)$,

iff $A\rho \in lub(T_P^{\mathcal{X}}(X))$. □

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iff $(A \leftarrow c|A_1, \dots, A_n) \in P$, $\mathcal{X} \models c\rho$ and $\{A_1\rho, \dots, A_n\rho\} \subset I$,
for some $I \in X$ (as X is a chain)
iff $A\rho \in T_P^{\mathcal{X}}(I)$ for some $I \in X$, iff $A\rho \in lub(T_P^{\mathcal{X}}(X))$. □

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$T_P^{\mathcal{X}}$ admits a least (post) fixed point $T_P^{\mathcal{X}} \uparrow \omega$

Full abstraction

Theorem 11 ([JL87popl])

$T_P^{\mathcal{X}} \uparrow \omega = O_{gs}(P)$

$T_P^{\mathcal{X}} \uparrow \omega \subset O_{gs}(P)$ is proved by induction on the powers n of $T_P^{\mathcal{X}}$.

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$n = 0$, i.e., $\emptyset$, is trivial.

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$O_{gs}(P) \subset T_P^{\mathcal{X}} \uparrow \omega$ is proved by induction on the length of derivations.

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$T_P^{\mathcal{X}} \uparrow \omega \subset O_{gs}(P)$ is proved by induction on the powers n of $T_P^{\mathcal{X}}$. $n = 0$, i.e., $\emptyset$, is trivial. Let $A\rho \in T_P^{\mathcal{X}} \uparrow n$, there exists a rule $(A \leftarrow c|A_1, \ldots, A_n) \in P$, s.t. $\{A_1\rho, \ldots, A_n\rho\} \subset T_P^{\mathcal{X}} \uparrow n-1$ and $\mathcal{X} \models c\rho$. By induction $\{A_1\rho, \ldots, A_n\rho\} \subset O_{gs}(P)$. By definition of O_{gs} and $\wedge$-compositionality. we get $A\rho \in O_{gs}(P)$.

$O_{gs}(P) \subset T_P^{\mathcal{X}} \uparrow \omega$ is proved by induction on the length of derivations. Successes with derivation of length 0 are ground facts in $T_P^{\mathcal{X}} \uparrow 1$.

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$O_{gs}(P) \subset T_P^{\mathcal{X}} \uparrow \omega$ is proved by induction on the length of derivations. Successes with derivation of length 0 are ground facts in $T_P^{\mathcal{X}} \uparrow 1$.
Let $A\rho \in O_{gs}(P)$ with a derivation of length n. By definition of O_{gs} there exists

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$O_{gs}(P) \subset T_P^{\mathcal{X}} \uparrow \omega$ is proved by induction on the length of derivations. Successes with derivation of length 0 are ground facts in $T_P^{\mathcal{X}} \uparrow 1$. Let $A\rho \in O_{gs}(P)$ with a derivation of length n. By definition of O_{gs} there exists $(A \leftarrow c|A_1, \ldots, A_n) \in P$ s.t. $\{A_1\rho, \ldots, A_n\rho\} \subset O_{gs}(P)$ and $\mathcal{X} \models c\rho$.

Full abstraction

Theorem 11 ([JL87popl])

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$T_P^{\mathcal{X}}$ and $\mathcal{X}$-models

Proposition 12

I is a $\mathcal{X}$-model of P iff I is a post-fixed point of $T_P^{\mathcal{X}}$, $T_P^{\mathcal{X}}(I) \subset I$

Proof.

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$T_P^{\mathcal{X}}$ and $\mathcal{X}$-models

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Proof.

I is a $\mathcal{X}$-model of P,
iff for each clause $A \leftarrow c|A_1, \ldots, A_n \in P$ and for each $\mathcal{X}$-valuation ρ, if $\mathcal{X} \models c\rho$ and $\{A_1\rho, \ldots, A_n\rho\} \subset I$ then $A\rho \in I$,
iff $T_P^{\mathcal{X}}(I) \subset I$ □

$T_P^{\mathcal{X}}$ and $\mathcal{X}$-models

Theorem 13 (Least $\mathcal{X}$-model [JL87popl])

Let P be a constraint logic program on $\mathcal{X}$. P has a least $\mathcal{X}$-model, denoted by $M_P^{\mathcal{X}}$ satisfying:

$$M_P^{\mathcal{X}} = T_P^{\mathcal{X}} \uparrow \omega$$

Proof.

$T_P^{\mathcal{X}} \uparrow \omega = lfp(T_P^{\mathcal{X}})$ is also the least post-fixed point of $T_P^{\mathcal{X}}$, thus by Prop. 12, $lfp(T_P^{\mathcal{X}})$ is the least $\mathcal{X}$-model of P. □

Fixpoint semantics of O_{ca}

Consider the set of constrained atoms
$\mathcal{B}'_{\mathcal{X}} = \{c|A \ : \ A$ is an atom and $\mathcal{X} \models \exists(c)\}$ modulo renaming

Consider the lattice of constrained interpretations $(2^{\mathcal{B}'_{\mathcal{X}}}, \subset)$

For a constrained interpretation I, let us define the closed $\mathcal{X}$-interpretation:
$[I]_{\mathcal{X}} = \{A\rho \ :$ there exists a valuation ρ and $c|A \in I$ s.t. $\mathcal{X} \models c\rho\}$

Let us define the semantics $O_{ca}(P)$ as the least solution of a fixpoint equation over $2^{\mathcal{B}'_{\mathcal{X}}}$

Non-ground immediate consequence operator

$S_P^{\mathcal{X}} : 2^{\mathcal{B}'_{\mathcal{X}}} \to 2^{\mathcal{B}'_{\mathcal{X}}}$ is defined as:

$S_P^{\mathcal{X}}(I) =$ $\{c|A \in \mathcal{B}'_{\mathcal{X}} \mid$ there exists a renamed clause in normal form $(A \leftarrow d|A_1, \ldots, A_n) \in P$, and constrained atoms $\{c_1|A_1, \ldots, c_n|A_n\} \subset I$, s.t. $c = d \wedge \bigwedge_{i=1}^{n} c_i$ is $\mathcal{X}$-satisfiable$\}$

Proposition 14

For any $\mathcal{B}'_{\mathcal{X}}$-interpretation I, $[S_P^{\mathcal{X}}(I)]_{\mathcal{X}} = T_P^{\mathcal{X}}([I]_{\mathcal{X}})$

Proof.

$A\rho \in [S_P^{\mathcal{X}}(I)]_{\mathcal{X}}$

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$\{A_1\rho, \ldots, A_n\rho\} \subset [I]_{\mathcal{X}}$
iff $A\rho \in T_P^{\mathcal{X}}([I]_{\mathcal{X}})$ □

Continuity of $S_P^{\mathcal{X}}$ operator

Proposition 15

$S_P^{\mathcal{X}}$ is continuous

Proof.

Continuity of $S_P^{\mathcal{X}}$ operator

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$S_P^{\mathcal{X}}$ *is continuous*

Proof.

Let X be a chain of constrained interpretations. $c|A \in S_P^{\mathcal{X}}(lub(X))$, iff $(A \leftarrow d|A_1, \ldots, A_n) \in P$, $c = d \wedge \bigwedge_{i=1}^{n} c_i$, $\mathcal{X} \models \exists(c)$ and $\{c_1|A_1, \ldots, c_n|A_n\} \subset lub(X)$

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iff $c|A \in S_P^{\mathcal{X}}(I)$ for some $I \in X$,
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Corollary 16

Continuity of $S_P^{\mathcal{X}}$ operator

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iff $c|A \in S_P^{\mathcal{X}}(I)$ for some $I \in X$,
iff $c|A \in lub(S_P^{\mathcal{X}}(X))$ □

Corollary 16

$S_P^{\mathcal{X}}$ *admits a least (post) fixed point* $lfp(S_P^{\mathcal{X}}) = S_P^{\mathcal{X}} \uparrow \omega$

Example CLP($\mathcal{H}$)

```
append(A,B,C):- A=[], B=C.
append(A,B,C):- A=[X|L], C=[X|R], append(L,B,R).
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Example 17

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\begin{aligned}
S_P^{\mathcal{H}} \uparrow 0 &= \emptyset \\
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&\quad \{A = [X|L], C = [X|R], L = [], B = R \mid append(A,B,C)\} \\
&= S_P^{\mathcal{H}} \uparrow 1 \cup \{A = [X], C = [X|B] \mid append(A,B,C)\} \\
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S_P^{\mathcal{H}} \uparrow 3 &= S_P^{\mathcal{H}} \uparrow 2 \ \cup \\
&\quad \{A = [X, Y], C = [X, Y|B] \mid append(A, B, C)\} \\
S_P^{\mathcal{H}} \uparrow 4 &= S_P^{\mathcal{H}} \uparrow 3 \ \cup
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S_P^{\mathcal{H}} \uparrow 4 &= S_P^{\mathcal{H}} \uparrow 3 \ \cup \\
&\quad \{A = [X, Y, Z], C = [X, Y, Z|B] \mid append(A, B, C)\} \\
\ldots &= \ldots
\end{aligned}
$$

Relating $S_P^{\mathcal{X}}$ and $T_P^{\mathcal{X}}$ operators

Theorem 18 ([JL87popl])

For every ordinal α, $T_P^{\mathcal{X}} \uparrow \alpha = [S_P^{\mathcal{X}} \uparrow \alpha]_{\mathcal{X}}$

Proof.

The base case $\alpha = 0$ is trivial. For a successor ordinal, we have

$$\begin{array}{lcl} [S_P^{\mathcal{X}} \uparrow \alpha]_{\mathcal{X}} & = & [S_P^{\mathcal{X}}(S_P^{\mathcal{X}} \uparrow \alpha - 1)]_{\mathcal{X}} \\ & = & \end{array}$$

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For a limit ordinal, we have

$$\begin{array}{rcl} [S_P^{\mathcal{X}} \uparrow \alpha]_{\mathcal{X}} & = & [\bigcup_{\beta<\alpha} S_P^{\mathcal{X}} \uparrow \beta]_{\mathcal{X}} \\ & = & \bigcup_{\beta<\alpha} [S_P^{\mathcal{X}} \uparrow \beta]_{\mathcal{X}} \\ & = & \bigcup_{\beta<\alpha} T_P^{\mathcal{X}} \uparrow \beta \text{ by induction} \\ & = & T_P^{\mathcal{X}} \uparrow \alpha \end{array}$$

□

Full abstraction w.r.t. computed answers

Theorem 19 (Theorem of full abstraction [GL91iclp])

$O_{ca}(P) = S_P^{\mathcal{X}} \uparrow \omega$

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Program analysis by abstract interpretation

$S_P^{\mathcal{H}} \uparrow \omega$ captures the set of computed answer constraints nevertheless this set may be infinite and may contain too much information for proving some properties of the computed constraints

Abstract interpretation [CC77popl] is a method for proving properties of programs without handling irrelevant information

The idea is to replace the real computation domain by an abstract computation domain which retains sufficient information w.r.t. the property to prove

Groundness analysis by abstract interpretation

Consider the CLP($\mathcal{H}$) append program

```
append(A,B,C):- A=[], B=C.
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What is the groundness relation between arguments after a success?

The term structure can be abstracted by a boolean structure which expresses the groundness of the arguments.
We thus associate a CLP($\mathcal{B}$) abstract program:

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We thus associate a CLP($\mathcal{B}$) abstract program:

```
append(A,B,C):- A=true, B=C.
append(A,B,C):- A=X/\L, C=X/\R, append(L,B,R).
```

Its least fixed point computed in at most 2^3 steps will express the groundness relation between arguments of the concrete program.

Groundness analysis (continued)

$$S_P^{\mathcal{B}} \uparrow 0 = \emptyset$$
$$S_P^{\mathcal{B}} \uparrow 1 =$$

Groundness analysis (continued)

$$
\begin{aligned}
S_P^{\mathcal{B}} \uparrow 0 &= \emptyset \\
S_P^{\mathcal{B}} \uparrow 1 &= \{A = \mathit{true}, B = C \mid \mathit{append}(A, B, C)\} \\
S_P^{\mathcal{B}} \uparrow 2 &= S_P^{\mathcal{B}} \uparrow 1 \ \cup
\end{aligned}
$$

Groundness analysis (continued)

$$
\begin{aligned}
S_P^{\mathcal{B}} \uparrow 0 &= \emptyset \\
S_P^{\mathcal{B}} \uparrow 1 &= \{A = true, B = C \mid append(A, B, C)\} \\
S_P^{\mathcal{B}} \uparrow 2 &= S_P^{\mathcal{B}} \uparrow 1 \ \cup \\
&\quad \{A = X \wedge L, C = X \wedge R, L = true, B = R \mid append(A, B, C)\} \\
&= S_P^{\mathcal{B}} \uparrow 1 \cup \{C = A \wedge B \mid append(A, B, C)\} \\
S_P^{\mathcal{B}} \uparrow 3 &= S_P^{\mathcal{B}} \uparrow 2 \ \cup
\end{aligned}
$$

Groundness analysis (continued)

$$
\begin{aligned}
S_P^{\mathcal{B}} \uparrow 0 &= \emptyset \\
S_P^{\mathcal{B}} \uparrow 1 &= \{A = \mathit{true}, B = C \mid \mathit{append}(A, B, C)\} \\
S_P^{\mathcal{B}} \uparrow 2 &= S_P^{\mathcal{B}} \uparrow 1 \ \cup \\
&\quad \{A = X \wedge L, C = X \wedge R, L = \mathit{true}, B = R \mid \mathit{append}(A, B, C)\} \\
&= S_P^{\mathcal{B}} \uparrow 1 \cup \{C = A \wedge B \mid \mathit{append}(A, B, C)\} \\
S_P^{\mathcal{B}} \uparrow 3 &= S_P^{\mathcal{B}} \uparrow 2 \ \cup \\
&\quad \{A = X \wedge L, C = X \wedge R, R = L \wedge B \mid \mathit{append}(A, B, C)\} \\
&= S_P^{\mathcal{B}} \uparrow 2 \cup \{C = A \wedge B \mid \mathit{append}(A, B, C)\} \\
&= S_P^{\mathcal{B}} \uparrow 2 = S_P^{\mathcal{B}} \uparrow \omega
\end{aligned}
$$

Groundness analysis (continued)

$$
\begin{aligned}
S_P^{\mathcal{B}} \uparrow 0 &= \emptyset \\
S_P^{\mathcal{B}} \uparrow 1 &= \{A = \mathit{true}, B = C \mid \mathit{append}(A, B, C)\} \\
S_P^{\mathcal{B}} \uparrow 2 &= S_P^{\mathcal{B}} \uparrow 1 \ \cup \\
&\quad \{A = X \wedge L, C = X \wedge R, L = \mathit{true}, B = R \mid \mathit{append}(A, B, C)\} \\
&= S_P^{\mathcal{B}} \uparrow 1 \cup \{C = A \wedge B \mid \mathit{append}(A, B, C)\} \\
S_P^{\mathcal{B}} \uparrow 3 &= S_P^{\mathcal{B}} \uparrow 2 \ \cup \\
&\quad \{A = X \wedge L, C = X \wedge R, R = L \wedge B \mid \mathit{append}(A, B, C)\} \\
&= S_P^{\mathcal{B}} \uparrow 2 \cup \{C = A \wedge B \mid \mathit{append}(A, B, C)\} \\
&= S_P^{\mathcal{B}} \uparrow 2 = S_P^{\mathcal{B}} \uparrow \omega
\end{aligned}
$$

In a success of $\mathit{append}(A, B, C)$,
C is ground iff A and B are ground.

Groundness analysis of reverse

Concrete CLP($\mathcal{H}$) program:

```
rev(A,B) :- A=[], B=[].
rev(A,B) :- A=[X|L], rev(L,K), append(K,[X],B).
```

Abstract CLP($\mathcal{B}$) program:

Groundness analysis of reverse

Concrete CLP($\mathcal{H}$) program:

```
rev(A,B) :- A=[], B=[].
rev(A,B) :- A=[X|L], rev(L,K), append(K,[X],B).
```

Abstract CLP($\mathcal{B}$) program:

```
rev(A,B) :- A=true, B=true.
rev(A,B) :- A=X/\L, rev(L,K), append(K,X,B).
```

Groundness analysis of reverse

Concrete CLP($\mathcal{H}$) program:

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rev(A,B) :- A=[], B=[].
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Abstract CLP($\mathcal{B}$) program:

```
rev(A,B) :- A=true, B=true.
rev(A,B) :- A=X/\L, rev(L,K), append(K,X,B).
```

$$
\begin{aligned}
S_P^{\mathcal{B}} \uparrow 0 &= \emptyset \\
S_P^{\mathcal{B}} \uparrow 1 &= \{A = true, B = true \mid rev(A,B)\} \\
S_P^{\mathcal{B}} \uparrow 2 &= S_P^{\mathcal{B}} \uparrow 1 \cup \{A = X, B = X \mid rev(A,B)\} \\
&= S_P^{\mathcal{B}} \uparrow 1 \cup \{A = B \mid rev(A,B)\} \\
S_P^{\mathcal{B}} \uparrow 3 &= S_P^{\mathcal{B}} \uparrow 2 \cup \{A = X \wedge L, L = K, B = K \wedge X \mid rev(A,B)\} \\
&= S_P^{\mathcal{B}} \uparrow 2 \cup \{A = B \mid rev(A,B)\} = S_P^{\mathcal{B}} \uparrow 2 = S_P^{\mathcal{B}} \uparrow \omega
\end{aligned}
$$

Constraint-based Model Checking [DP99tacas]

Analysis of unbounded states concurrent systems by CLP programs.
Concurrent transition systems defined by condition-action rules [Shankar93acm]:

$$\text{condition } \phi(\vec{x}) \text{ action } \vec{x}' = \psi(\vec{x})$$

Translation into CLP clauses over one predicate p (for states)

$$p(\vec{x}) \leftarrow \phi(\vec{x}),\ \psi(\vec{x}', \vec{x}),\ p(\vec{x}').$$

The transitions of the concurrent system are in one-to-one correspondance to the CSLD derivations of the CLP program.

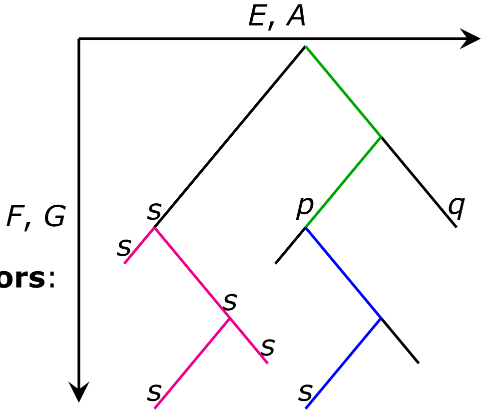
Proposition 20

The set of states from which a set of states defined by a constraint c is reachable is the set $lfp(T_P)$ where P is the CLP program plus the clause $p(\vec{x}) \leftarrow c(\vec{x})$.

Computation Tree Logic CTL

Temporal logic for branching time:

- States described by propositional or first-order formulas
- Two **path quantifiers** for non-determinism:
  - *A* “for all paths”
  - *E* “for some path”
- Several **temporal operators**:
  - *X* “next time”,
  - *F* “eventually”,
  - *G* “always”,
  - *U* “until”.



Model Checking

Two types of interesting properties:
$AG\neg\phi$ "Safety" property.
$AF\psi$ "Liveness" property.

Duality: for any formula ϕ we have
$EF\phi = \neg AG\neg\phi$ and
$EG\phi = \neg AF\neg\phi$.

Model checking is an algorithm for computing, in a given Kripke structure $K = (S, I, R)$, $I \subset S, R \subset S \times S$ (S is the set of states, I the initial states and R the transition relation), the set of states which satisfy a given CTL formula ϕ, i.e., the set $\{s \in S | K, s \models \phi\}$.

(Symbolic) Model Checking

Basic algorithm

When S is finite, represent K as a graph, and iteratively label the nodes with the subformulas of ϕ which are true in that node.

Add A to the states satisfying A ($\neg A$, $A \wedge B$, ...)

Add $EF\phi$ ($EX\phi$) to the (immediate) predecessors of states labeled by ϕ

Add $E(\phi U \psi)$ to the predecessor states of ψ while they satisfy ϕ

Add $EG\phi$ to the states for which there exists a path leading to a non trivial strongly connected components of the subgraph restricted to the states satisfying ϕ

Symbolic model checking

Use OBDD's to represent states and transitions as boolean formulas (S is finite).

Constraint-based Model Checking

Constraint-based model checking [DP99tacas] applies to Kripke structures with an infinite set of states.
Numerical constraints provide a finite representation for an infinite set of states.

Constraint logic programming theory:

$$EF(\phi) = lfp(T_{R \cup \{p(\vec{x}) \leftarrow \phi\}})$$

$$EG(\phi) = gfp(T_{R \wedge \phi})$$

Prototype implementation *DMC* in Sicstus Prolog $+$ Simplex, CLP($\mathcal{H}, \mathcal{FD}, \mathcal{R}, \mathcal{B}$)