Constraint Grammars

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Outline

1. Introduction.


4. Conclusions & Perspectives.
Constraint Sets.

Observational confluence

\[
\text{nil}(L) \setminus \text{positive_list}(L) \iff \text{true}.
\]
\[
\text{list}(L, H, T) \setminus \text{positive_list}(L) \iff H > 0, \text{positive_list}(T).
\]

- Confluent for queries modelling a list
  (first arguments of list tokens are pair-wise distincts).
- Non-confluent for queries modelling a tree
  (with two list tokens sharing a common first argument).

Characterizing (L)CC Stores

Describing Global Constraints

Given a set of constraints \( S \):

- either there exists a satisfied constraint in \( S \):
  \[
  \text{REGULAR}(x_1,\ldots,x_n,L) \text{ with } S = \{ x_1 = u_1 \land \cdots \land x_n = u_n \}_{u \in L}
  \]

- or all the constraints of \( S \) are satisfied:
  \[
  \text{ALL-DIFFERENT}(x_1,\ldots,x_n) \text{ with } S = \{ x_i \neq x_j \}_{i \neq j}
  \]
Linear Constraint Systems.

Let $V$ be a set of variables.

- If $C$ is a set of formulae over the set of variables $V$, let $\mathcal{C}[C]$ be the closure of $C$ by conjunction ($\otimes$) and variable hiding ($\exists$).
- If $\vdash$ is a binary relation over $C$, let $\hat{\vdash}$ be the closure of $\vdash$ by the rules of linear logic.

**Definition**

A (linear) **constraint system** is a pair $(C, \vdash)$ where:

- $C$ is a set of formulae closed by conjunction ($\otimes$) and variable hiding ($\exists$);
- $\vdash \subseteq C^2$ is a relation closed by the rules of linear logic.
Examples of Constraint Systems

Let $\Sigma$ be a signature. Let $t(\Sigma) \doteq \{f(v_1, \ldots, v_n) | f / n \in \Sigma\}$.

**linear-token system**

The **linear-token system** over $\Sigma$ is the constraint system $\text{TOK}(\Sigma) \doteq (\mathcal{C}[t(\Sigma)], \emptyset)$.

**linear-token system with equality**

The **linear-token system with equality** over $\Sigma$ is the constraint system $\text{TOK}_=(\Sigma) \doteq (\mathcal{C}[t(\Sigma)] \cup \{(x = y | x, y \in V)\}, \vdash)$, where $\vdash$ is the smallest relation with the axioms of the equality theory:

$\vdash !(x = x); !(x = y) \vdash !(y = x); !(x = y) \otimes !(y = z) \vdash !(x = z)$;

$f(x_0, \ldots, x_n) \otimes !(x_0 = y_0) \otimes \cdots \otimes !(x_n = y_n) \vdash f(y_0, \ldots, y_n)$
Context-Free Constraint Grammars (CFCG).

**Definition**

A context-free constraint grammar (CFCG) is a tuple \((V, C, \vdash, \Sigma, P)\) where:

- \(V\) is a set of variables and \((C, \vdash)\) is a constraint-system over \(V\);
- \(\Sigma\) is the signature of non-terminal symbols:
  \[ N \doteq \{ f(y_1, \ldots, y_n) \mid f/n \in \Sigma \text{ and } y_1, \ldots, y_n \in V \text{ pair-wise distincts} \} \]
- \(P \subseteq N \times \overline{C}\) is the set of productions, with \(\overline{C} \doteq \overline{C}[C \cup N]\).
  Every production \((f(y_1, \ldots, y_n), u) \in P\) is such that \(\text{fv}(u) \subseteq \{y_1, \ldots, y_n\}\).
  A production is denoted: \(f(y_1, \ldots, y_n) ::= u\).

**Example:**

\[
\begin{align*}
h(X, Y) & ::= \text{edge}(X, Y) \\
h(X, Y) & ::= \exists Z.\text{edge}(X, Z) \otimes h(Z, Y)
\end{align*}
\]
Derivations for CFCGs.

Let \((V, C, \vdash, \Sigma, P)\) be a context-free constraint grammar. Let \(\overline{C} \equiv \mathcal{C}[C \cup N]\).

**Definition**

Let \(\rightarrow \subseteq \overline{C}^2\) be the smallest relation satisfying the following rules:

\[
\begin{align*}
  f(y_1, \ldots, y_n) &\equiv u \\
  (f(y_1, \ldots, y_n))\sigma &\rightarrow (u)\sigma \\
  u \rightarrow v &\Rightarrow \exists x. u \rightarrow \exists x v \\
  u \otimes w &\rightarrow v \otimes w \\
  w \otimes u &\rightarrow w \otimes v
\end{align*}
\]

A derivation is a sequence \((c_i)_i\) with elements in \(\overline{C}\) such that \(c_0 \rightarrow c_1 \rightarrow \ldots\).

**Example:**

\[
\begin{align*}
  h(X, Y) &\equiv \text{edge}(X, Y) \\
  h(X, Y) &\equiv \exists Z. \text{edge}(X, Z) \otimes h(Z, Y) \\
  h(A, B) &\rightarrow \exists X_1. \text{edge}(A, X_1) \otimes h(X_1, B) \\
  &\rightarrow \exists X_1 X_2. \text{edge}(A, X_1) \otimes \text{edge}(X_1, X_2) \otimes h(X_2, B) \\
  &\rightarrow \exists X_1 X_2. \text{edge}(A, X_1) \otimes \text{edge}(X_1, X_2) \otimes \text{edge}(X_2, B)
\end{align*}
\]
Languages accepted by CFCGs.

Let $G = (V, C, \vdash, \Sigma, P)$ be a context-free constraint grammar. Let $\overline{C} = \overline{C}[C \cup N]$.

**Definition**

The **constraint language** accepted by a schema $c_0 \in \overline{C}$ and the grammar $G$ is:

$$L_G(c_0) = \{ c \in C | (\exists n) (\exists (c_i)_{0 \leq i \leq n}) (c_0 \rightarrow \cdots \rightarrow c_n \land c_n \vdash \vdash c) \}$$

**Example:**

$$h(X, Y) ::= \text{edge}(X, Y)$$

$$h(X, Y) ::= \exists Z. \text{edge}(X, Z) \otimes h(Z, Y)$$

$$L_G(h(A, B)) = \{ \text{hamiltonian paths from } A \text{ to } B \}$$
Claim

Over linear token systems, context-free constraint grammars have the same expressive power than hyperedge replacement graph grammars.


Theorem (K.-J. Lange and E. Welzl 87)

*Hyperedge replacement graph language membership problem is NP-complete.*
Hypergraphs.

A **labelled set** $\dot{E}$ is a tuple $(E, E^\Sigma, E^\lambda)$, with $E^\Sigma$ alphabet and $E^\lambda : E \rightarrow E^\Sigma$.

**Definition**

A **hypergraph** is a tuple $(V, \dot{E}, S, \rightarrow)$, where:

- $V$ is the set of vertices; $\dot{E}$ is the labelled set of hyperedges;
- $S$ is the alphabet of selectors;
- $\rightarrow \subseteq E \times S \times V$ is the incidence relation. $(e, s, v) \in \rightarrow$ is denoted $e \overset{s}{\rightarrow} v$.
- $\rightarrow$ is functional: $(\forall e s v_1 v_2) e \overset{s}{\rightarrow} v_1 \land e \overset{s}{\rightarrow} v_2 \Rightarrow v_1 = v_2$

Vertices are represented as fat dots, hyperedges as boxes.

Let $\mathcal{H}(V, E, S)$ the set of hypergraphs over $V, E, S$. 

![Hypergraph Example](image-url)
Embedding of Hypergraphs into Constraints.

Let $H = (V, \dot{E}, S, \rightarrow)$ be a hypergraph.

- For all $e \in E$, let $\text{dom}(e) = \{ s \in S | (\exists v \in V)(e \overset{s}{\rightarrow} v) \}$.
- Let $n(e) = |\text{dom}(e)|$ and $r_e : \{1, \ldots, n(e)\} \leftrightarrow \text{dom}(e)$.
- Let $\Sigma = \{ f / n \in E^\Sigma \times \mathbb{N} | (\exists e \in E)(f = E^\lambda(e) \text{ and } n = n(e)) \}$.

**Definition**

The **constraint-embedding** of the hypergraph $h$ is the following constraint of $\text{TOK}(\Sigma)$:

\[
\llbracket h \rrbracket \overset{\text{def}}{=} (\exists v_1 \ldots v_n) \left( \bigotimes_{e \in E \land n=n(e) \land y_i=r_e(i)} [E^\lambda(e)](y_1, \ldots, y_n) \right)
\]

**Proposition**

For every closed constraint $c \in \text{TOK}(\Sigma)$, there exists a hypergraph $h$ such that

\[\llbracket h \rrbracket \vdash c\]
Hypergraph Isomorphism.

Definition

Two hypergraphs $h_0 \doteq (V, (E, E^\Sigma, E^\lambda_0), S, \rightarrow_0)$ and $h_1 \doteq (V, (E, E^\Sigma, E^\lambda_1), S, \rightarrow_1)$ are **isomorphic**, denoted $h_0 \cong h_1$, when there exist $\sigma_V : V \leftrightarrow V$ and $\sigma_E : E \leftrightarrow E$ such that:

- $(\forall e \in E)(E^\lambda_0(e) = E^\lambda_1(\sigma_E(e)))$;
- $(\forall e \in E)(\forall s \in S)(\forall v \in V)(e \xrightarrow{s_0} v \iff (\sigma_E(e) \xrightarrow{s_1} \sigma_V(v)))$.

Proposition

$$h_0 \cong h_1 \iff \llbracket h_0 \rrbracket \vdash \llbracket h_1 \rrbracket$$
Hyperedge Replacement Graph Grammars (HRGs).

Let $N, V, E, E^\Sigma, S$ be defined accordingly to the following definition.

We denote $\mathcal{H}^+(N, V, E, E^\Sigma, S)$ the set of **augmented hypergraphs**, that is hypergraphs whose edges are labelled with $E^\Sigma$ and vertices are in $\hat{V}$.

**Definition**

A **h-edge replacement graph grammar** is a tuple $(N, V, E, E^\Sigma, S, P)$, with:

- $N$ is the alphabet of non-terminals, such that $E^\Sigma \cap N = \emptyset$; $\hat{E}^\Sigma = E^\Sigma \cup N$;
- $V$ is the set of vertices;
- $E$ is the set of hyperedges and $E^\Sigma$ the set of ground labels;
- $S$ is the alphabet of selectors, such that $V \cap S = \emptyset$; $\hat{V} = V \cup S$;
- $P \subseteq N \times \wp(S) \times \mathcal{H}^+(N, V, E, E^\Sigma, S)$ is the set of productions.
Derivations for HRGs.

Let \((N, V, E, E^\Sigma, S, P)\) be a hyperedge replacement graph grammar.

**Definition**

Let \(\rightarrow \subseteq \mathcal{H}^+(N, V, E, E^\Sigma, S)^2\) be the relation such that:

\((V_0, \hat{E}_0, S, \rightarrow_0) \rightarrow (V_1, \hat{E}_1, S, \rightarrow_1)\) iff there exist:

- a hyperedge \(e \in E_0\); a production \((n, S_p, (V_p, \hat{E}_p, S, \rightarrow_p)) \in P\);
- \(\sigma^E_{0 \rightarrow 1} : E_0 \setminus \{e\} \leftarrow E_1\), \(\sigma^V_{0 \rightarrow 1} : V_0 \leftarrow V_1\), \(\sigma^E_{p \rightarrow 1} : E_p \leftarrow E_1\), \(\sigma^V_{p \rightarrow 1} : V_p \setminus S \leftarrow V_1\)

such that:

- \(E^\lambda_0(e) = n\); \(\text{im}(\sigma^E_{0 \rightarrow 1}) \cup \text{im}(\sigma^E_{p \rightarrow 1}) = E_1\) and \(\text{im}(\sigma^V_{0 \rightarrow 1}) \cup \text{im}(\sigma^V_{p \rightarrow 1}) = V_1\);
- \((\forall e_0 \in E_0 \setminus \{e\}) (\forall e_p \in E_p) (\forall s, s' \in S) (\forall v_0 \in V_0) (\forall v_p \in V_p \setminus S)\)

\[
E^\lambda_0(e_0) = E^\lambda_1(\sigma^E_{0 \rightarrow 1}(e_0)) \quad \quad \quad \quad e_0 \xrightarrow{s_0} v_0 \iff \sigma^E_{0 \rightarrow 1}(e_0) \xrightarrow{s_1} \sigma^V_{0 \rightarrow 1}(v_0)
\]

\[
E^\lambda_p(e_p) = E^\lambda_1(\sigma^E_{p \rightarrow 1}(e_p)) \quad \quad \quad \quad e_p \xrightarrow{s_p} v_p \iff \sigma^E_{p \rightarrow 1}(e_p) \xrightarrow{s_1} \sigma^V_{p \rightarrow 1}(v_p)
\]

\[
e \xrightarrow{s_0} v_0 \land e_p \xrightarrow{s'_p} s \iff \sigma^E_{p \rightarrow 1}(e_p) \xrightarrow{s'_1} \sigma^V_{0 \rightarrow 1}(v_0)
\]
Embedding of HRGs into CFCGs.

Let $G=(N, V, E, E^\Sigma, S, P)$ be a hyperedge replacement graph grammar.

1. $\bullet$ can be extended to an embedding of augmented hypergraphs into scheme of $C$, with $N=\{f(y_1, \ldots, y_n)| (\exists h)(f, \{y_1, \ldots, y_n\}, h) \in P\}$;
2. therefore $\bullet$ can be extended to an embedding of HRG productions into context-free constraint productions;
3. therefore $\bullet$ can be extended to an embedding of HRG grammars into context-free constraint grammars.

**Definition**

The **hypergraph language** accepted by an augmented hypergraph $h_0 \in \mathcal{H}^+(N, V, E, E^\Sigma, S)$ and the grammar $G$ is:

$$L_G(h_0) = \{h \in \mathcal{H}(V, E, S)| (\exists n)(\exists (h_i)_{0 \leq i \leq n})(h_0 \rightarrow \cdots \rightarrow h_n \land h \simeq h_n)\}$$

**Proposition**

For all $h_0 \in \mathcal{H}^+(N, V, E, E^\Sigma, S)$, we have $[L_G(h)] = L_{[G]}([h])$. 
With Equality Theory.

**Lemma:** for all constraint $c \in \text{TOK}_= (\Sigma)$, there exists a constraint $c' \in \text{TOK}(\Sigma) \subseteq \text{TOK}_= (\Sigma)$ such that $c \vdash \vdash c'$.

**Proposition**

For every closed constraint $c \in \text{TOK}_= (\Sigma)$, there exists a hypergraph $h$ such that $[h] \vdash \vdash c$

**Lemma:** For a schema $c \in \text{TOK}_= (\Sigma)$ and a grammar $G$ over $\text{TOK}_= (\Sigma)$, there exists $c' \in \text{TOK}(\Sigma)$ and a $G'$ over $\text{TOK}(\Sigma)$ such that:

$$L_G(c) = \{ h \in \text{TOK}_= (\Sigma) | \exists h' \in L_{G'}(c'), h \vdash \vdash h' \}$$

**Proposition**

Over $\text{TOK}_= (\Sigma)$, context-free constraint language membership problem is NP-complete.
Applications to Global Constraints.

Let $A$ be an alphabet. Every $u = u_1 \cdots u_n \in A^*$ can be represented as a string-hypergraph:

$$
\begin{array}{cccc}
0 & u_1 & 1 & \cdots & 0 & u_n & 1 & 0
\end{array}
$$

**Proposition (Habel and Kreowski 87)**

*Every context-free string language $G$ can be recognized by a hyperedge replacement graph grammar $[G]$.*

**Corollary:** Every context-free string language $G$ can be recognized by a context-free constraint grammar $[G]$.

**Proposition**

$$\text{CONTEXT-FREE}(x_1, \ldots, x_n, G) \equiv (\exists) [G] (x_1, \ldots, x_n)$$
Limitations.

Definition

A **jungle** is a hypergraph \((V, \dot{E}, \{\text{in}_k, \text{out}\}, \rightarrow)\) such that:

- \((\forall v \in V) \left(\left|\left\{ e \in E \mid e \overset{\text{out}}{\rightarrow} v \right\}\right| \leq 1\right);\)
- \(\rightarrow\) is acyclic.


“**Corollary 2:** CHR folding is terminating and confluent.”

*Only true if the query is a jungle!*

Proposition

\(\{ h \in \mathcal{H}(V, E, S) \mid h \text{ is a jungle} \}\) is not a hyperedge replacement graph language.
A two-sorted signature $\Sigma$ is a subset of $F \times \mathbb{N} \times \mathbb{N}$, where $F$ is a countable symbol set. Items $(f, n, m) \in \Sigma$ are denoted $f / (n, m)$.

Let $V$ and $W$ be two set of variables and $L$ a set of labels. The set constraint system is the smallest classical constraint system $(S, \vdash_S)$ such that:

\[
\begin{align*}
&\frac{w \in W}{w = \emptyset \in S} & \quad \frac{v \in V \quad w \in W}{w = \{v\} \in S} & \quad \frac{v \in V \quad w \in W}{(v \in w) \in S} & \quad \frac{w \in W \quad l \in L}{w = fv l \in S} \\
&\frac{w_1 \in W \quad w_2 \in W \quad w_3 \in W}{w_1 = w_2 \; op \; w_3 \in S} & \quad op \in \{\cap, \cup, \setminus\}
\end{align*}
\]

with $w = \{v\} \vdash_S v \in w$  
$w = \emptyset \vdash_S v \notin w$

and $v \in w_1 \; op_1 \quad v \in w_2 \vdash_S v \in w_1 \; op_2 \quad w_2$ with

$(op_1, op_2) \in \{(\land, \cap); (\lor, \cup); (\land \neg, \cap)\}$...
A domain-sensitive constraint grammar (DSCG) is a tuple $(V, W, L, C, \vdash, \Sigma, P)$ where:

- $V, W$ are sets of variables, $L$ a set of labels;
- $(C, \vdash)$ is a constraint-system over $V$;
- $\Sigma$ is the two-sorted signature of non-terminal symbols:
  \[
  N \doteq \left\{ l : f(y_1, \ldots, y_n, Y_1, \ldots, Y_m) \mid l \in L, f/\langle n, m \rangle \in \Sigma \text{ and } y_1, \ldots, y_n \in V, Y_1, \ldots, Y_m \in W \right\}
  \]
  p.-w. dist.
- $P \subseteq N \times \overline{C} \times S$ is the set of productions, with $\overline{C} \doteq C \cup N$.

A production is denoted: $f(y_1, \ldots, y_n, Y_1, \ldots, Y_m) ::= u|g$. 
Jungles  Let $\Sigma$ be a signature. For all $f/n \in \Sigma$:

$$j(r) ::= \text{eq}(r, f(x_1, \ldots, x_n)) \otimes l_1 : j(x_1) \otimes \cdots \otimes l_n : j(x_n)|r \not\in \text{fv}(l_1) \cup \cdots \cup \text{fv}(l_n)$$

all-different

$$a(S) ::= x_1 \neq x_2|x_1 \in S \land x_2 \in S \setminus \{x_1\}$$

$$\text{ALL}-\text{DIFFERENT}(x_1, \ldots x_n) \equiv (\forall) a(\{x_1, \ldots, x_n\})$$
Conclusions & Perspectives.

Objective

- Describing set of constraints;
- for observational confluence;
- for defining global constraints.

Contributions

- Definition of constraint grammar;
- as expressive as hyperedge-replacement graph grammar;
- much more easy to define;
- more general.

Future work

- Decidability and complexity analysis for other constraint systems.