Games on strings
with a limited ordering

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GAMES 2008
September 8th - 11th, 2008
Warsaw, Poland
Outline of the presentation

- Basics on EF-Games
- Remoteness
- Labeled $<_p$ structures
- Local games on $<_p$ structures
- Local games on labeled $<_p$ structures
- Global games on labeled $<_p$ structures
- Efficient algorithm to compute remoteness
- Conclusions and future work
EF-Games

- (Logical) combinatorial games
- The playground: two relational structures $\mathcal{A}$ and $\mathcal{B}$ (over the same finite vocabulary)
- Two players: Spoiler and Duplicator
- Move by Spoiler: select a structure and pick an element in it
- Move by Duplicator: pick an element in the opposite structure
- Round: a move by Spoiler followed by a move by Duplicator
- Game: sequence of rounds
- Duplicator tries to imitate Spoiler
Winning strategies

- **Configuration:** \(((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b})), \text{ with } |\vec{a}| = |\vec{b}|\)
  - Represents the relation \(\{(a_i, b_i) | 1 \leq i \leq |\vec{a}|\}\)

- **A play** from \(((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b})))\) proceeds by extending the initial configuration with the pair of elements chosen by the two players, e.g.
  - if Spoiler picks \(c\) in \(\mathcal{A}\),
  - and Duplicator replies with \(d\) in \(\mathcal{B}\),
  - then the new configuration is \(((\mathcal{A}, \vec{a}, c), (\mathcal{B}, \vec{b}, d))\)

- **Ending condition:** a player repeats a move or the configuration is not a partial isomorphism

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**Definition**

Duplicator has a **winning strategy** from \(((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b})))\) if every configuration of the game until an ending configuration is reached is a partial isomorphism, no matter how Spoiler plays.
An example on graphs

- Duplicator must respect the adjacency relation...
- ...and pick nodes with the same label as Spoiler does
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- Duplicator must respect the adjacency relation...
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• Duplicator must respect the adjacency relation. . .
• . . . and pick nodes with the same label as Spoiler does
Bounded and unbounded games

- **Bounded game**: $G_m(A, B)
- The number $m$ of rounds is fixed
- The game ends after $m$ rounds have been played
- **Unbounded game**: $G(A, B)$
- The game goes on as long as either a player repeats a move, or the current configuration in not partial isomorphism
- Duplicator wins iff the final configuration is a partial isomorphism
Winning and optimal strategies

Winning strategy ≠ Optimal strategy

• In unbounded EF-games, Spoiler wins unless $A \simeq B$
• “Play randomly” is a winning strategy for Spoiler
• But, how far actually is the end of a game?
• What are the best (optimal) moves?
• Remoteness of $G$: the minimum $m$ such that Spoiler wins $G_m$
  • Simplified definition under the hypothesis $A \not\cong B$
• Optimal Spoiler’s move: whatever Duplicator replies, the remoteness decreases
• Optimal Duplicator’s move: no matter how Spoiler has played, the remoteness decreases at most by 1
Main uses of EF-games

- Prove inexpressibility results (Ehrenfeucht’s theorem)
- Establish normal forms for logics (Gaifman’s theorem)
- Prove elementary equivalence (Hanf’s theorem) and \(m\)-equivalence (Sphere lemma) of structures
- Determine how and where two structures differ: use of remoteness to measure the degree of similarity between two structures

Our aim

Compare biological sequences
A simple example

Consider the following two sequences:

\[
\text{agggagtttttaga} \quad \text{agtttagtttagaagggga}
\]

The standard left-to-right comparison:

\[
\begin{align*}
\text{a} & \quad \text{g} & \quad \text{g} & \quad \text{g} & \quad \text{a} & \quad \text{g} & \quad \text{t} & \quad \text{t} & \quad \text{t} & \quad \text{a} & \quad - & \quad - & \quad - & \quad - & \quad \text{g} & \quad \text{a} \\
\mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid & \quad \mid \\
\text{a} & \quad \text{g} & \quad \text{t} & \quad \text{t} & \quad \text{a} & \quad \text{g} & \quad \text{t} & \quad \text{t} & \quad \text{t} & \quad \text{a} & \quad \text{g} & \quad \text{a} & \quad \text{a} & \quad \text{g} & \quad \text{g} & \quad \text{g} & \quad \text{g} & \quad \text{a} & \quad \text{a}
\end{align*}
\]

A more flexible way of comparing sequences:

\[
\begin{align*}
\text{agggagtttttaga} \\
\text{agtttagtttagaagggga}
\end{align*}
\]
A simple example

Consider the following two sequences:

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\]

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\[
\begin{array}{ccccccccccccccccc}
\text{a} & \text{g} & \text{g} & \text{g} & \text{a} & \text{g} & \text{t} & \text{t} & \text{t} & \text{t} & \text{a} & \text{g} & \text{g} & \text{a} \\
\| & \| & \| & \| & \| & \| & \| & \| & \| & \|
\end{array}
\]

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\[
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A simple example

Consider the following two sequences:

```
agggagtttttaga  agtttagtttagaagggga
```

The standard left-to-right comparison:

```
<table>
<thead>
<tr>
<th>a</th>
<th>g</th>
<th>g</th>
<th>g</th>
<th>a</th>
<th>g</th>
<th>t</th>
<th>t</th>
<th>t</th>
<th>t</th>
<th>t</th>
<th>a</th>
</tr>
</thead>
</table>

A more flexible way of comparing sequences:

```
agggagtttttaga  agtttagtttagaagggga
```

Labeled successor structures

Definition

Let $\Sigma$ be a fixed alphabet and $w \in \Sigma^*$. A *labeled successor structure* is a pair $(w, i^n)$ where

- $w = (\{1, \ldots, |w|\}, \text{succ}, (P_a)_{a \in \Sigma})$
- $(i, j) \in \text{succ}$ iff $j = i + 1$ for all $i, j \in \{1, \ldots, |w|\}$
- $i \in P_a$ iff $w[i] = a$ for all $i \in \{1, \ldots, |w|\}$
- $i^n$ are distinguished positions $i_1, \ldots, i_n \in \{1, \ldots, |w|\}$

- Necessary and sufficient conditions for Duplicator to win $G_q((w, i^n), (w', j^n))$
- Computation of remoteness in polynomial time using suffix trees (LPAR 2005, GAMES 2007)
The relation $<_p$

What about the linear order relation $<$?

Locality is destroyed :(.

We introduce a limited order relation ($<_p$) that lies in between the successor and the linear order relations:

$$i <_p j \text{ iff } i < j \text{ and } j - i \leq p$$

The successor relation and the linear order relation are recovered as special cases of the limited order relation for $p = 1$ and $p = \infty$, respectively.
The relation \(<_p\)

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Our contribution

- Necessary and sufficient conditions for Duplicator to win $G_q((w, i^n), (w', j^n))$ on labeled $<_p$ structures
- Algorithm to compute the remoteness in polynomial time

Local and global strategy

- **Local strategy**: how Duplicator must reply when Spoiler plays in the neighborhoods of already selected positions
- **Global strategy**: how Duplicator must reply when Spoiler plays far from already selected positions
Local games on \(<_p\) structures: \(p\text{step}\)-safety

\(p\text{step}\): “signed distance” between two positions in terms of the number of intervals of length \(p\) separating them

Let \(i, j, k, p \in \mathbb{N}\), with \(i, j, p > 0\) and \(k \geq p\).

\[
p\text{step}_{k}^{(p)}(i, j) = \begin{cases} 
0 & \text{if } i = j \\
\left\lceil \frac{i - j}{p} \right\rceil & \text{if } |i - j| \leq k \text{ and } i < j \\
\left\lfloor \frac{i - j}{p} \right\rfloor & \text{if } |i - j| \leq k \text{ and } i > j \\
\infty & \text{if } |i - j| > k 
\end{cases}
\]

A configuration \((w, w', i^n, j^n)\) is \(p\text{step-safe}\) in the \(k\)-horizon if

\(p\text{step}_{k}^{(p)}(i_r, i_s) = p\text{step}_{k}^{(p)}(j_r, j_s)\) for all \(r, s \in \{1 \ldots n\}\)

Lemma

Let \(w, w' \in \Sigma^*\). If \((w, w', i^n, j^n)\) is not \(p\text{step-safe}\) in the \((p \cdot 2^q)\)-horizon, then Spoiler wins \(G_q((w, i^n), (w', j^n))\).
Example of *pstep*-safety

**Figure:** *pstep*-safety.
Local games on $<_p$ structures: $\theta$-safety

$\vartheta_k$: truncated signed distance between two positions

Let $i, j, k \in \mathbb{N}$, with $i, j, k > 0$.

\[
\vartheta_k(i, j) = \begin{cases} 
  i - j & \text{if } |i - j| \leq k \\
  \infty & \text{otherwise}
\end{cases}
\]

A configuration $(w, w', i^n, j^n)$ is $\vartheta$-safe in the $k$-horizon if $\vartheta_k(i_r, i_s) = \vartheta_k(j_r, j_s)$ for all $r, s \in \{1 \ldots n\}$

**Lemma**

Let $w, w' \in \Sigma^*$ and $q > 0$. If $(w, w', i^n, j^n)$ is not $\vartheta$-safe in the $(2^q - 1)$-horizon, then Spoiler wins $G_q((w, i^n), (w', j^n))$. 
Rigid and elastic intervals

- The neighborhood of each position can be partitioned in rigid and elastic intervals (each position origins $2^{q-2} + 1$ right and $2^{q-2} + 1$ left $q$-rigid intervals)

- 0th $q$-rigid interval induced by $i$:
  $$\rho_{0,q}^+(i) = \rho_{0,q}^-(i) = [i - \alpha_q^0, i + \alpha_q^0],$$
  where $\alpha_q^0 = 2^{q-1} - 1$

- $k$th right $q$-rigid interval induced by $i$, with $0 < k \leq 2^{q-2}$:
  $$\rho_{k,q}^+(i) = (c - \alpha_q^z, c + \alpha_q^z],$$
  where $c = i + kp$ and $\alpha_q^z$ depends on $q$ and on $z = \lceil \log_2 k \rceil + 1$.

$q = 5; \ 2^{q-2} = 8$
Local games on $<_p$ structures: $p$-int-safety

**Definition**

Let $q > 0$. A configuration $(w, w', i^n, j^n)$ is \textit{p-int-safe} in the $k$-horizon if for all $r, s \in \{1, \ldots, n\}$, with $r < s$, if there exists $0 \leq h \leq 2^{k-1}$ such that $i_s \in \rho^+_{h,k+1}(i_r)$ or $j_s \in \rho^+_{h,k+1}(j_r)$, then $i_s - i_r = j_s - j_r$.

**Lemma**

Let $w, w' \in \Sigma^*$ and $q > 0$. If $(w, w', i^n, j^n)$ is not $p$-int-safe in the $q$-horizon, then Spoiler wins $G_q((w, i^n), (w', j^n))$.

**Remark**

If $(w, w', i^n, j^n)$ is $p$-int-safe in the $q$-horizon, with $q > 0$, then it is $\vartheta$-safe in the $(2^q - 1)$-horizon.
Local games on labeled $<_p$ structures

**Definition**

Let $w \in \Sigma^*$, $q, p \in \mathbb{N}$, with $p > 1$, and $i \in \mathbb{Z}$. The $q$-color of position $i$ in $w$, denoted by $q\text{-}col_w(i)$, is inductively defined as follows:

- the 0-color of $i$ in $w$ is the label $w[i]$;
- the $(q+1)$-color of $i$ in $w$ is the label $w[i]$ plus the $q$-color of each of the $2^q$ right intervals and of the $2^q$ left intervals induced by $i$.

The $q$-color of the $j$th right interval $[a, b]$ induced by $i$, with $1 \leq j \leq 2^q$, is the ordered tuple

$$t^w_a \ldots t^w_{a+\gamma_1-1} \{ t^w_{a+\gamma_1} \ldots t^w_{b-\gamma_2} \} t^w_{b-\gamma_2+1} \ldots t^w_b,$$

where for all $a \leq i \leq b$, $t^w_i =$q\text{-}col_w(i)$ and $\gamma_1$ and $\gamma_2$ depend on the radius of rigid intervals.
<p>-safety for q-colors

**Definition**

Let \( w, w' \in \Sigma^* \) and \( p, n, q \in \mathbb{N} \), with \( p > 0 \). A configuration \((w, w', i^n, j^n)\) is \(<_p\)-safe for q-colors if

for all \( r \in \{1, \ldots, n\} \), q-col\(_w(i_r) = \)q-col\(_w'(j_r)\)

**Lemma**

Let \( w, w' \in \Sigma^* \), and \( p, q \in \mathbb{N} \), with \( p > 1 \). If \((w, w', i^n, j^n)\) is not \(<_p\)-safe for q-colors, then Spoiler wins \( G_q((w, i^n), (w', j^n)) \).
Example of $<_p$-safety for $q$-colors

$q = 2; \Sigma = \{a, b\}; \ p = 10$

Figure: Safety for $q$-colors.
Main result (for the local case)

Definition

A configuration \((w, w', i^n, j^n)\) is \textit{q-locally-safe} if it is \textit{pstep-safe} in the \((p \cdot 2^q)\)-horizon, \textit{p-int-safe} in the \(q\)-horizon, and \(<_p\text{-safe}\) for \(q\)-colors.

Theorem

\textbf{[Sufficient condition for Duplicator to win]}

Let \(w, \ w' \in \Sigma^*\), and \(p, q \in \mathbb{N}\), with \(p > 1\). If \((w, w', i^n, j^n)\) is \textit{q-locally-safe}, then Duplicator wins \(G_q((w, i^n), (w', j^n))\).
Global games on labeled $<_p$ structures

The two strings must have the same $q$-colors and, for each color, the same multiplicity and a similar distribution.

Let $q, p \in \mathbb{N}^+$, $i^n$ be a set of positions in $w$ and $\tau$ be a $(q - 1)$-color.

- $P_{(q,p)}^{(w,i^n)} = \{j \mid (q - 1)\text{-color}_w(j) = \tau \land j \text{ falls “far” from } i^n\}$
- $q$-multiplicity: $\rho_{(q,p)}^{(w,i^n)}(\tau) = |P_{(q,p)}^{(w,i^n)}|$
- $k$-scattered set $S$: $|a - b| > k$ for all $a, b \in S$
- $q$-scattering $\sigma_{(q,p)}^{(w,i^n)}(\tau)$: maximal cardinality of a $(p2^q)$-scattered subset of $P_{(q,p)}^{(w,i^n)}$
- $\Delta_{(q,p)}^{(w,i^n)}(w',j^n) = \{\tau \mid \tau \text{ is a (q-1)-color, } q > 0, \text{ and } \sigma_{(q,p)}^{(w,i^n)}(\tau) \neq \sigma_{(q,p)}^{(w',i^n)}(\tau) \lor \rho_{(q,p)}^{(w,i^n)}(\tau) \neq \rho_{(q,p)}^{(w',i^n)}(\tau)\}.$
Main result (for the global case)

Theorem

[Main Theorem]
Let \( w, w' \in \Sigma^* \) and \( p, q \in \mathbb{N} \), with \( p > 1 \). Duplicator wins \( G_q((w, i^n), (w', j^n)) \) if and only if the following conditions hold:

1. \((w, w', i^n, j^n)\) is \( q \)-locally-safe;

2. for all \((r-1)\)-color \( \tau \in \Delta^{(w, i^n)}_{(w', j^n)} \), with \( 1 \leq r \leq q \),
   \[
   \sigma^{(w, i^n)}_{(i, p)}(\tau) > q - r \quad \text{and} \quad \sigma^{(w', j^n)}_{(i, p)}(\tau) > q - r.
   \]

Remoteness of \( G \): \( r + \min(\sigma^{(w, i^n)}_{(r, p)}, \sigma^{(w', j^n)}_{(r, p)}) \).
Complexity of remoteness

- Compute in polynomial time scattering and multiplicity of a $q$-color in a string ($O(p^2 n^3 \log n)$)
- Compare in polynomial time two $q$-colors ($O(p^2 n^3 \log n)$)
- Each $q$-color is represented by a layered directed graph
- Bottom-up visit of the graphs
Conclusions and future work

• We analyzed *EF-games* on labeled $<_p$ structures.

• We identified necessary and sufficient winning conditions for Spoiler and Duplicator, that allow one to compute the remoteness of a game and optimal strategies for both players.

• **Next step**: extensive experimentation of the proposed games on real biological data.
THANKS
Basic definitions

- **Vocabulary**: finite set of relation symbols
- $\mathcal{A}$ and $\mathcal{B}$ structures on the same vocabulary
- $\vec{a} = a_1, \ldots, a_k \in \text{dom}(\mathcal{A})$
- $\vec{b} = b_1, \ldots, b_k \in \text{dom}(\mathcal{B})$
- **Configuration**: $((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$, with $|\vec{a}| = |\vec{b}|$
  - Represents the relation $\{(a_i, b_i) \mid 1 \leq i \leq |\vec{a}|\}$

**Definition**

$((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$ is a **partial isomorphism** if it is an isomorphism of the substructures induced by $\vec{a}$ and $\vec{b}$, respectively.
Main result

First-order EF-games capture \( m \)-equivalence

**Theorem (Ehrenfeucht, 1961)**

*Duplicator has a winning strategy in \( G_m((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b})) \) if and only if \((\mathcal{A}, \vec{a})\) and \((\mathcal{B}, \vec{b})\) satisfy the same FO-formulas of quantifier rank \( m \) and at most \(|\vec{a}|\) free variables, written \((\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})\).*

**Corollary**

*A class \( \mathcal{K} \) of structures (on the same finite vocabulary) is FO-definable if and only if there is \( m \in \mathbb{N} \) such that Spoiler has a winning strategy whenever \( \mathcal{A} \in \mathcal{K} \) and \( \mathcal{B} \notin \mathcal{K} \).*
Expressiveness results

Exploiting the corollary, we can prove negative expressiveness results.

Example

Let $\mathcal{L}_k \overset{\text{def}}{=} (\{1, \ldots, k\}, <)$. It is known that

$$n = p \text{ or } n, p \geq 2^m - 1 \Rightarrow \text{Duplicator wins } G_m(\mathcal{L}_n, \mathcal{L}_p)$$

“The class of linear orderings of even cardinality is not FO-definable”

- Given $m$, choose $\tilde{n} = 2^m$ and $\tilde{p} = 2^m + 1$;
- then, Duplicator wins $G_m(\mathcal{L}_{\tilde{n}}, \mathcal{L}_{\tilde{p}})$ (i.e., $\mathcal{L}_{\tilde{n}} \equiv_m \mathcal{L}_{\tilde{p}}$)
Example of $\vartheta$-safety

Figure: $\vartheta$-safety.
Rigid and elastic intervals

**Definition**

Let \( q > 1 \) and \( i \in \mathbb{N} \). The 0th \( q \)-rigid interval induced by position \( i \) is the closed interval \( \rho_{0,q}^+(i) = \rho_{0,q}^-(i) = [i - \alpha_0, i + \alpha_0] \), where \( \alpha_0 = 2^{q-1} - 1 \). The \( k \)th right (resp., left) \( q \)-rigid interval induced by position \( i \), with \( 0 < k \leq 2^{q-2} \), is the interval

\[
\rho_{k,q}^+(i) = (c - \alpha^z_q, c + \alpha^z_q) \quad \text{(resp.,} \quad \rho_{k,q}^-(i) = [c - \alpha^z_q, c + \alpha^z_q])
\]

where \( c = i + kp \) (resp., \( c = i - kp \)) and

\[
\alpha^z_q = 1 + \sum_{j=z-1}^{q-2} (2^j - 1), \quad \text{where} \quad z = \lceil \log_2 k \rceil + 1.
\]
Example of $p$-int-safety

(a) $q = 3$

(b) $q = 2$

(c) $q = 1$

(d) $q = 0$

Figure: $p$-int-safety.
Local games on labeled <_p structures

Definition

Let \( w \in \Sigma^* \), \( q, p \in \mathbb{N} \), with \( p > 1 \), and \( i \in \mathbb{Z} \). The \( q \)-color of position \( i \) in \( w \), denoted by \( q \text{-col}_w(i) \), is inductively defined as follows:

- the 0-color of \( i \) in \( w \) is the label \( w[i] \);
- the \((q+1)\)-color of \( i \) in \( w \) is the ordered tuple
  \[ \sigma_{2q}^w \cdots \sigma_1^w w[i] \tau_1^w \cdots \tau_{2q}^w \]
  where, for all \( 1 \leq j \leq 2^q \), \( \tau_j^w \) (resp., \( \sigma_j^w \)) is the \( q \)-color of the \( j \)-th right (resp., left) interval induced by \( i \).

The \textit{q-color} of the \( j \)-th right (resp., left) interval \([a, b]\) induced by \( i \), with \( 1 \leq j \leq 2^q \), is the ordered tuple

\[ t_a^w \cdots t_{a+\gamma_1-1}^w \{ t_{a+\gamma_1}^w \cdots t_{b-\gamma_2}^w \} t_{b-\gamma_2+1}^w \cdots t_b^w \] (resp.,
\[ t_a^w \cdots t_{a+\gamma_2-1}^w \{ t_{a+\gamma_2}^w \cdots t_{b-\gamma_1}^w \} t_{b-\gamma_1+1}^w \cdots t_b^w \)),

where for all \( a \leq i \leq b \), \( t_i^w = q \text{-col}_w(i) \) and \( \gamma_1 \) and \( \gamma_2 \) depend on the radius of rigid intervals.
Example of safety for $q$-colors

$q = 2; \Sigma = \{a, b\}; \ p = 10$

**Figure:** Safety for $q$-colors.
• Let $n = |w| + |w'|$
• Multiplicity values can be computed in $O(n)$ time
• Scattering values can be computed in $O(n \log n)$ time
Global games on labeled $<_p$ structures (1)

The two strings must have the same $q$-colors and, for each color, the same multiplicity and a similar distribution.

Let $P \subseteq \mathbb{N}$ be a finite set. A \textit{k-blurred partition} $\mathcal{P}$ of $P$ is a partition of $P$ such that (i) for each $A \in \mathcal{P}$ and for each $a, b \in A$, $\delta(a, b) \leq k$, and (ii) there is not a partition $\mathcal{P}'$ satisfying (i) such that $|\mathcal{P}| > |\mathcal{P}'|$. The number of classes of $\mathcal{P}$ is called \textit{k-blurring}.

Let $q, p \in \mathbb{N}^+$, $i^n$ be a set of positions in $w$ and $\tau$ be a $(q-1)$-color.

\begin{itemize}
  \item $\rho^{(w, i^n)}_{(q, p)}(\tau)$: number of occurrences of $\tau$ which are “far” from $i^n$
  \item $\sigma^{(w, i^n)}_{(q, p)}(\tau)$: $(p2^q)$-blurring of occurrences of $\tau$ which are “far” from $i^n$
  \item $\Delta^{(w, i^n)}_{(w', j^n)} = \{\tau \mid \tau \text{ is a (q-1)-color, } q > 0, \text{ and } \sigma^{(w, i^n)}_{(q, p)}(\tau) \neq \sigma^{(w', j^n)}_{(q, p)}(\tau)\}$
\end{itemize}